Fourier Cosine and Sine transforms.

Fourier cosine transform is defined by pair

$$f_c^*(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx,$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c^*(\alpha) \cos \alpha x \, d\alpha.$$

Fourier sine transform is defined as

$$f_s^*(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx,$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s^*(\alpha) \sin \alpha x \, d\alpha.$$

Transforms of derivatives

(a) $\mathcal{F}_c\{f'(x)\} = \alpha f_c^*(\alpha) - \sqrt{\frac{2}{\pi}} f(0)$

(b) $\mathcal{F}_c\{f''(x)\} = -\alpha^2 f_c^*(\alpha) - \sqrt{\frac{2}{\pi}} f'(0)$

(c) $\mathcal{F}_s\{f'(x)\} = -\alpha f^*(\alpha)$

(d) $\mathcal{F}_s\{f''(x)\} = -\alpha^2 f^*(\alpha) + \sqrt{\frac{2}{\pi}} \alpha f(0)$

Convolution Theorem for Fourier Cosine and Sine Transforms

$$\int_0^\infty f_c^*(\alpha)\, g_c^*(\alpha) \cos \alpha x \, d\alpha = \frac{1}{2}\int_0^\infty f(u)\left[g(x+u) + g(|x-u|)\right] du \quad (1)$$

or $$\mathcal{F}_c^{-1}\left\{f_c^*(\alpha)\, g_c^*(\alpha)\right\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(u)\left[g(x+u) + g(|x-u|)\right] du.$$

Proof: Using the definition of Fourier inverse cosine transform

$$\mathcal{F}_c^{-1}\left\{f_c^*(\alpha)\, g_c^*(\alpha)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f_c^*(\alpha)\, g_c^*(\alpha) \cos x\alpha \, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty g_c^*(\alpha) \cos \alpha x \, d\alpha \int_0^\infty f(u) \cos \alpha u \, du$$

$$= \frac{2}{\pi}\int_0^\infty f(u)\, du \int_0^\infty \cos \alpha x \cos \alpha u \, g_c^*(\alpha)\, d\alpha$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\, du \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\cos \alpha(x+u) + \cos \alpha(|x-u|)\right] g_c^*(\alpha)\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty f(u)\left[g(x+u) + g(|x-u|)\right] du.$$

Similarly

$$\int_0^\infty f_s^*(\alpha)\, g_s^*(\alpha) \cos \alpha x \, d\alpha = \frac{1}{2}\int_0^\infty f(u)\left[g(u+x) + g(u-x)\right] du.$$

Parseval Relation Put $x = 0$ in (1)

$$\int_0^\infty f_c^*(\alpha)\, g_c^*(\alpha)\, d\alpha = \int_0^\infty f(u)\, g(u)\, du = \int_0^\infty f(x)\, g(x)\, dx.$$

If $g(x) = \overline{f(x)}$ then $g_c(\alpha) = \overline{f_c(\alpha)}$, so

$$\int_0^\infty |f_c^*(\alpha)|^2 \, d\alpha = \int_0^\infty |f(x)|^2 \, d\alpha.$$

Example (a) $\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \quad \frac{a}{a^2+\alpha^2}, \quad a>0$

(b) $\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \quad \frac{\alpha}{a^2+\alpha^2}, \quad a>0$

We have $\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos \alpha x \, dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\left[e^{i\alpha x} + e^{-i\alpha x}\right] dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty \left[e^{-(a-i\alpha)x} + e^{-(a+i\alpha)x}\right] dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{e^{-(a-i\alpha)x}}{-(a-i\alpha)} + \frac{e^{-(a+i\alpha)x}}{-(a-i\alpha)} \right]_0^\infty$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \left[\frac{1}{a-i\alpha} + \frac{1}{a+i\alpha} \right] = \sqrt{\frac{2}{\pi}} \quad \frac{a}{a^2+\alpha^2}.$$

Example PDE 1 $\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0, t>0$

$u(0,t) = 0, \quad u(x,0) = f(x), \; x>0.$

Fourier sine transform in x gives

$$\frac{dU_s^*}{dt} = -\alpha^2 u_s^*(\alpha,t)$$

$$\Rightarrow \quad u_s^*(\alpha,t) = A e^{-\alpha^2 t}.$$

Using $\mathcal{F}_s\{u(x,0) = f(x)\} \Rightarrow A = \sqrt{\frac{2}{\pi}} f_s^*(\alpha)$

so, $\quad u_s^*(\alpha,t) = \sqrt{\frac{2}{\pi}} f_s^*(\alpha) \, e^{-\alpha^2 t}$

$$\Rightarrow \quad u(x,t) = \frac{2}{\pi} \int^\infty f_s^*(\alpha) \, e^{-\alpha^2 t} \sin \alpha x \, d\alpha$$

Exercise: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x>0, t>0$

$u_x(0,t) = 0$, $u(x,0) = f(x)$, $x>0$.

Example PDE 2 - Laplace Eqn

$u_{xx} + u_{yy} = 0$, $\quad 0 < x, y < \infty$

$u(0,y) = a$, $\quad u(x,0) = 0$

$\nabla u \to 0$ as $r \to \infty$. $\quad a$ is constant

Using Fourier sine transform in x

$$\frac{d^2 u_s^*}{dy^2} - \alpha^2 u_s^* + \sqrt{\frac{2}{\pi}}\, \alpha a = 0$$

The solution to this inhomogeneous equation is

$$u_s^*(\alpha, y) = A e^{-\alpha y} + \sqrt{\frac{2}{\pi}}\ \frac{a}{\alpha}$$

As $\mathcal{F}_s\{u(x,0) = 0\} \Rightarrow u^*(\alpha, 0) = 0$, we get

$$A = -\sqrt{\frac{2}{\pi}}\ \frac{a}{\alpha}\ , \text{ so that}$$

$$u_s^*(\alpha, y) = \frac{a}{\alpha}\sqrt{\frac{2}{\pi}}\left(1 - e^{-\alpha y}\right)$$

The inverse transform gives

$$u(x,y) = \frac{2a}{\pi}\int_0^\infty \frac{1}{\alpha}\left(1 - e^{-\alpha y}\right) \sin \alpha x \, d\alpha \rightarrow \boxed{\frac{2a}{\pi}\tan^{-1}\left(\frac{x}{y}\right)}$$

We can simplify it using $\int_0^\infty e^{-a\alpha} \sin x\alpha \, d\alpha = \frac{x}{a^2+x^2}$

and integrating w r t a

$$\int_0^\infty \frac{e^{-a\alpha}}{\alpha} \sin x\alpha = \int_a^\infty \frac{x}{a^2+x^2}\, da = \left[\tan^{-1}\left(\frac{a}{x}\right)\right]_a^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{a}{x}\right) = \tan^{-1}\left(\frac{x}{a}\right)$$

Example PDE3 Laplace Eqn

$$u_{xx} + u_{yy} = 0 \quad ; \quad 0 < x < \infty \quad ; \quad 0 < y < b$$

$$u(0, y) = 0, \qquad u(x,y) \to 0 \quad \text{as } x \to \infty$$

$$u(x, b) = 0, \qquad u(x, 0) = f(x).$$

Fourier sine transform in $x \Rightarrow$

$$\frac{d^2 u_s^*}{dy^2} - \alpha^2 u_s^* = 0$$

$$u_s^*(\alpha, b) = 0, \qquad u_s^*(\alpha, 0) = f_s^*(\alpha).$$

The solution is

$$u_s^*(\alpha, y) \; f_s^*(\alpha) \; \frac{\sinh[\alpha(b-y)]}{\sinh \alpha b}$$

The inversion gives

$$u(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s^*(\alpha) \frac{\sinh[\alpha(b-y)]}{\sinh \alpha b} \sin \alpha x \, d\alpha$$