Zeros and Singularities:

Let $f(z)$ be function of complex variable z. A point z_0 is a "zero" of $f(z)$ if $f(z)$ is analytic in the NHD of z_0 and $f(z_0)=0$.

z_0 is an <u>isolated singularity</u> if f is analytic in $0<|z-z_0|<R$, for some R, but not analytic at z_0 itself.

If z_0 is an isolated singularity of $f(z)$, the Laurent series of $f(z)$ being

$$f(z) = \sum_{j=-\infty}^{+\infty} a_{+j}\,(z-z_0)^j$$

$$= \cdots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

(i) If $a_j = 0$ for all $j<0$, z_0 is a removable singularity. (Example: $f(z) = \frac{z-1}{z^2-1}$, $z_0 = +1$)

(ii) If $a_{-m} \neq 0$ (m +ve integer) but $a_j = 0$, $j<-m$ then z_0 is a <u>pole of order m</u> (Example: $f(z) = \frac{1}{(z-1)^2}$, $z=1$ pole of order 2)

(iii) If $a_j \neq 0$ for an infinite many negative values of j, z_0 is an essential singularity.

(Example: $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$
$z_0 = 0$ is an essential singularity).

Analytic Function.

$f(z)$ is analytic at z_0 if it has continuous derivatives of all orders at z_0. In that case, we can always write it as the Taylor series at $z = z_0$ i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

It may be noted that even if we use phrase '$f(z)$ is analytic at z_0' it is so in a NHD of z_0.

Line Integral If C is a curve given by $z = z(t)$, $a \le t \le b$, f continuous on C, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Integral along a Circle : Let C be a circle of radius R, then C: $z = Re^{i\theta}$, $0 \le \theta < 2\pi$

then $$\int_C f(z)\,dz = \int_0^{2\pi} f(Re^{i\theta}).\,Ri\,e^{i\theta}\,d\theta.$$

Consider $f(z) = z$, $C: z = Re^{i\theta}$, $0 \le \theta < 2\pi$

$$\int_C z\,dz = \int_0^{2\pi} Re^{i\theta}.\,R.i\,e^{i\theta}\,d\theta = R^2\int_0^{2\pi} e^{2i\theta}\,d\theta = 0.$$

Likewise $$\int_C z^n\,dz = R^{n+1}\int_0^{2\pi} e^{2i\theta}\,d\theta = 0, \quad n = 0, 1, 2, \dots$$

Then $$\int_C (1 + z + z^2 + \dots + z^n)\,dz = 0$$

where C is circle of radius R, centre 0.

Cauchy Integral Theorem

$$\oint_C f(z)\, dz = 0$$

where f is analytic in a simply connected domain enclosed by a simple closed curve C.

Again, consider a circle of radius R given by $z = R e^{i\theta}$, $0 \le \theta < 2\pi$.

$$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} R^{-1} e^{-i\theta} . i R e^{i\theta} d\theta = 2\pi i.$$

If $f(z)$ has a pole at $z=0$ then Laurent series will have the form

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

By above discussion, $C: z = Re^{i\theta}$, $0 \le \theta < 2\pi$

$$\oint_C f(z) dz = \int_C \left(\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots \right) dz$$

$$= 2\pi i \; a_{-1}.$$

a_{-1} is called the residue at $z=0$.

Cauchy Residue Theorem: Let f be analytic in a domain D except at $z_1, z_2, \ldots, z_n$ where f has poles.

Then $\displaystyle\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z_j} f.$

Calculation of Residue at at $z = z_0$

$\operatorname{Res}_{z=z_0} f$ is coefficient of $(z-z_0)^{-1}$ i.e. a_{-1}

For example $f(z) = \dfrac{\sin z}{z^2}$

$$= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = \frac{1}{z} - \frac{z}{3!} + \cdots$$

Thus $\operatorname{Res}_{z=0} \dfrac{\sin z}{z^2} = 1.$

Also, $\operatorname{Res}\left(e^{\frac{1}{z}}\right) = 1$

Residue at a Simple pole.

If z_0 is a simple pole of $f(z)$ then

$$\operatorname{Res}_{z=z_0} f = \lim_{z\to z_0} \frac{f(z)}{z-z_0}.$$

Residue at a pole of order m

If z_0 is a pole of order m

$$\operatorname{Res}_{z=z_0} f = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

Example. $f(z) = \dfrac{\cos z}{(z+i)^3}$

$z = -i$ is pole of order 3.

$$\operatorname{Res}_{z=-i} f = \frac{1}{2!} \lim_{z\to -i} \frac{d^2}{dz^2}\left[(z+i)^3 \frac{\cos z}{(z+i)^3}\right]$$

$$= \lim_{z\to -i} \frac{1}{2} \frac{d^2}{dz^2} \cos z = -\frac{1}{2}\cos i.$$

Example Residue Theorem

Compute $\oint_\Gamma f(z)\,dz$ where $f(z) = \dfrac{iz - \cos z}{z^3+z}$

and Γ is a closed path enclosing $0, \pm i$.

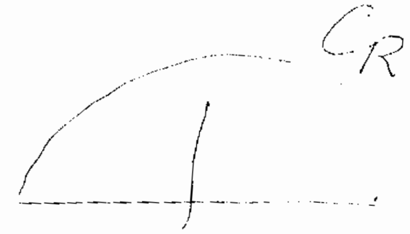
Ans. $2\pi i\,[$

A simple Rule

If $f(z) = \dfrac{p(z)}{q(z)}$ and z_0 is a pole then

$$\operatorname*{Res}_{z=z_0} f = \frac{p(z_0)}{q'(z_0)} \quad \text{(Use L'Hopital rule)}$$

Jordan's Lemma

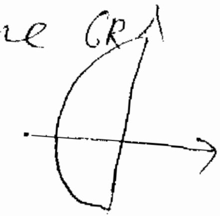


Suppose we have a circular arc C_R with center 0, $f(z) \to 0$ uniformly as $R \to \infty$. Then

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{\pm imz}\, dz = 0 \quad , \quad m > 0$$

if C_R is in upper half plane / lower half plane

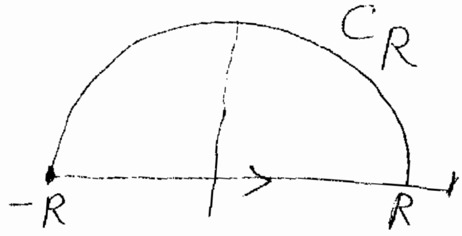
$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{\cancel{\pm} mz}\, dz = 0, \quad m > 0$$



if C_R is in left half plane / right half plane

Application.

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2}\, dx$



Consider $\displaystyle\int_{\Gamma} \frac{e^{ikz}}{z^2+a^2}\, dz$

$$= \int_{-R}^{R} \frac{e^{ikx}}{x^2+a^2}\, dx + \int_{C_R} \frac{e^{ikz}}{z^2+a^2}\, dz$$

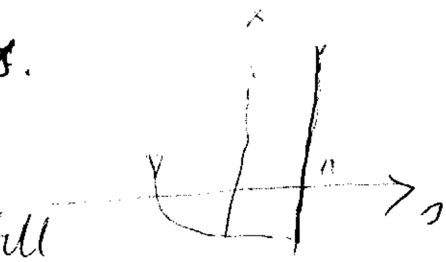
By Jordan's Lemma $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{z^2+a^2}\, dz \to 0$

$$\int_{-R}^{R} \frac{e^{ikx}}{x^2+a^2}\, dx \to \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2}\, dx \quad \text{as } R \to \infty$$

Evaluation of inverse Laplace transform

We shall see that if Laplace transform of $f(x)$ is defined as $\mathcal{L}\{f(x)\} = \int_0^\infty f(x)\, e^{-sx}\, dx = F(s)$

then $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(s)\, e^{sx}\, ds.$



The form of the Jordan Lemma will require that our contour is in the left half-plane (as shown here). The integral on the curved part of the contour, under conditions of Jordan's Lemma vanishes as $R \to \infty$. In that case:

$$\frac{1}{2\pi i} \int_{\Gamma} F(z)\, e^{zx}\, dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F(s)\, e^{sx}\, ds$$

$$= \frac{1}{2\pi i} \left(2\pi i \sum_k \operatorname{Res}_{z_k} F(z) e^{xz} \right)$$

$$= \sum_k \operatorname{Res}_{z_k} F(z)\, e^{xz}.$$

In other words, the Laplace inversion of $F(s)$ is given by sum of residues of $F(z)\, e^{xz}$.

Example Find $\mathcal{L}^{-1}\{F(s)\}$, $F(s) = \frac{1}{s^2+a^2}$

$$F(z)\, e^{xz} = \frac{1}{z^2+a^2}\, e^{xz}.$$

Poles are at $z = \pm i a$

$$\operatorname*{Res}_{z=ia} F(z)\,e^{xz} = \lim_{z\to ia} \frac{\cancel{(z-ia)}\; e^{xz}}{(z+ia)\cancel{(z-ia)}}$$

$$= \frac{1}{2ia}\, e^{iax}$$

$$\operatorname*{Res}_{z=-ia} F(z)\,e^{xz} = \lim_{z\to -ia} \frac{\cancel{(z+ia)}}{\cancel{(z+ia)}(z-ia)}\, e^{xz}$$

$$= -\frac{1}{2ia}\, e^{-iax}$$

$$\sum \operatorname{Res} F(z)\,e^{xz} = \frac{1}{a}\,\frac{e^{iax} - e^{-iax}}{2i} = \frac{1}{a}\sin ax.$$

Examples Find inverse Laplace transform

(a) $F(s) = \dfrac{1}{(s-2)^2(s+4)}$ (b) $F(s) = \dfrac{1}{s^4+1}$.

Bromwich Theorem (1916)

$$f(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} F(z)\, e^{xz}\, dz$$

where $F(z)$ is the Laplace transform

$$F(s) = \int_0^\infty f(x)\, e^{-sx}\, dx.$$

c is the real number such that above L.T is well defined for $s > c$.