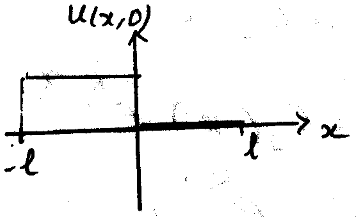
)

PDE $\quad u_t = k\, u_{xx} \quad , \; -l < x < l$

B.C $\quad u(-l,t) = u(l,t)$

$\quad u_x(-l,t) = u_x(l,t)$

I.C. $\quad u(x,0) = \begin{cases} 0, & 0 < x < l \\ 1 & -l < x < 0 \end{cases}$



Step 1 $\quad$ Assume $\quad u(x,t) = F(x)\, G(t)$

So, $\quad u_{xx} = F''(x)\, G(t)$

$\quad u_t = F(x)\, G'(t)$

PDE $\Rightarrow$ After usual steps are argument

$$\frac{F''(x)}{F(x)} = \frac{1}{k}\,\frac{G'(t)}{G(t)} = \lambda \text{ (say)}$$

B.C: $\quad u(-l,t) = u(l,t) \Rightarrow F(-l)G(t) = F(l)G(t)$

$\Rightarrow F(-l) = F(l) \quad$ — (a)

$u_x(-l,t) = u(l,t) \Rightarrow F'(-l)G(t) = F'(l)G(t)$

$\Rightarrow F'(-l) = F'(l) \quad$ — (b)

(a) and (b) are separated boundary conditions. The corresponding D.E is

$$F''(x) = \lambda F(x)$$

$\lambda = 0$: $\quad$ In this case $F(x) = Ax + B$

$F(-l) = F(l) \Rightarrow \quad -Al + \cancel{B} = Al + \cancel{B}$

$\Rightarrow \quad A = 0$

Thus $\quad F(x) = B$ (arbitrary constant) corresponding to $\lambda = 0$.

$\lambda > 0$ In this case take $\lambda = k^2$, k real.

Then $F'' - k^2 F = 0$

The auxiliary equation: $m^2 - k^2 = 0$

$m = \pm k$

The general solution

$$F(x) = c_1 e^{kx} + c_2 e^{-kx}$$

also $F'(x) = c_1 k e^{kx} + c_2(-k) e^{-kx}$

$$= k(c_1 e^{kx} - c_2 e^{-kx})$$

$F(-l) = F(l) \Rightarrow c_1 e^{kl} + c_2 e^{-kl} = c_1 e^{-kl} + c_2 e^{kl}$

or $c_1(e^{kl} - e^{-kl}) + c_2(e^{-kl} - e^{kl}) = 0$ —— (b1)

$F'(-l) = F'(l) \Rightarrow c_1 e^{kl} - c_2 e^{-kl} = c_1 e^{-kl} - c_2 e^{kl}$

or $c_1(e^{kl} - e^{-kl}) + c_2(e^{kl} - e^{-kl}) = 0$ —— (b2)

(b1) and (b2) will have nontrivial solutions in c_1 and c_2 if det(coeff) = 0 i.e.

$$\begin{vmatrix} e^{kl} - e^{-kl} & e^{-kl} - e^{+kl} \\ e^{kl} - e^{-kl} & e^{kl} - e^{-kl} \end{vmatrix} = (e^{kl} - e^{-kl})^2 + (e^{kl} - e^{-kl})^2$$

$$= 0$$

$\Rightarrow e^{kl} - e^{-kl} = 0 \Rightarrow k = 0$ as in above

case

$\lambda < 0$. For this $\lambda = -k^2$, k = real

The auxiliary equation is $m^2 + k^2 = 0$

or $m^2 = \pm ik$

The general solution is

$$F(x) = C_1 \cos kx + C_2 \sin kx$$

$$F'(x) = -kC_1 \sin kx + kC_2 \cos kx$$

$F(l) = F(-l) \implies C_1 \cos kl + C_2 \sin kl$

$$= C_1 \cos kl - C_2 \sin kl$$

$\implies 2C_2 \sin kl \implies 0$ or $\sin kl = 0$ $(C_2 \neq 0)$

This gives $kl = n\pi$ or $k = \frac{n\pi}{l}$, $n = 1, 2, 3 \ldots$

$F'(l) = F'(-l)$ will also lead to

$$-kC_1 \sin kl + kC_2 \cancel{\cos kl} = kC_1 \sin kl + \cancel{kC_2 \cos kl}$$

$$\implies k = \frac{n\pi}{l}.$$

As $F(x) = C_1 \cos kx + C_2 \sin kx$

C_1, C_2 arbitrary, we get two linearly independent eigenfunctions corresponding to $k_n = \frac{n\pi}{l}$ $\left(\lambda_n = -\frac{n^2\pi^2}{l^2}\right)$ (label by n)

$$\left.\begin{aligned} &\text{For, } C_1 = 1, C_2 = 0 \quad & F_n^{(1)} &= \cos \frac{n\pi}{l} x \\ &\text{For } C_1 = 0, C_2 = 1, \quad & F_n^{(2)} &= \sin \frac{n\pi}{l} x \end{aligned}\right\} n = 1, 2, 3 \ldots$$

Thus we have $\boxed{F_n = A_0 \cdot 1 + A_n \cos \frac{n\pi}{l} x + B_n \sin \frac{n\pi}{l} x.}$

$n = 1, 2, 3 \ldots$

or $F_n = A_n \cos \frac{n\pi}{l} x + B_n \sin \frac{n\pi}{l} x$

$n = 0, 1, 2, 3, \ldots$

Step 2 $\quad \frac{1}{k}\frac{G'(t)}{G(t)} = \lambda = -\frac{n^2\pi^2}{\ell^2}$ gives

$$G'(t) = -\frac{n^2\pi^2}{\ell^2}k$$

or $\frac{dG}{dt} = -\frac{n^2\pi^2}{\ell^2}k$

which has solution

$$G_n(t) = C_n e^{-\frac{n^2\pi^2}{\ell^2}kt} \qquad n = 1,2,3 \ldots$$

Step 3: $\quad u_n(x,t) = F_n(x)\, G_n(t)$

$$= A_0 + \left(\tilde{A}_n \cos\left(\tfrac{n\pi}{\ell}x\right) + \tilde{B}_n \sin\left(\tfrac{n\pi}{\ell}x\right)\right) e^{-\frac{n^2\pi^2}{\ell^2}kt}$$

By principle of superposition, $\left(\tilde{A}_n = C_n A_n,\ \tilde{B}_n = C_n B_n\right)$

$$u(x,t) = \sum_n u_n(x,t)$$

$$= A_0 + \sum_{n=1}^{\infty}\left[\tilde{A}_n \cos\left(\tfrac{n\pi x}{\ell}\right) + \tilde{B}_n \sin\left(\tfrac{n\pi x}{\ell}\right)\right] e^{-\frac{n^2\pi^2}{\ell^2}kt}$$

Step 4: $\quad$ Determination of $A_0, \tilde{A}_n, \tilde{B}_n$.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} \tilde{A}_n \cos\left(\tfrac{n\pi x}{\ell}\right) + \tilde{B}_n \sin\left(\tfrac{n\pi x}{\ell}\right)$$

$$= f(x) = \begin{cases} 0, & 0<x<\ell \\ 1, & -\ell<x<0 \end{cases}$$

Integrate both sides w.r.t. x from $-\ell$ to ℓ

$$\int_{-\ell}^{\ell} A_0\, dx = 2\ell A_0 = \int_{-\ell}^{\ell} f(x)\, dx = \int_{-\ell}^{0} dx + \int_{0}^{\ell} 0 . dx$$

$$\boxed{A_0 = \tfrac{1}{2}} \qquad = [x]_{-\ell}^{0} = \ell.$$

To determine $\tilde{A}_n$, multiply throughout ⑤
by $\cos\left(\frac{m\pi x}{l}\right)$ and integrate from $-l$ to l

$$\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx = A_0\int_{-l}^{l}\cos\frac{m\pi x}{l}\,dx$$

$$+\sum_{n=1}^{\infty}\tilde{A}_n\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx$$

$$+\tilde{B}_n\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx$$

But $\int_{-l}^{l}\cos\frac{m\pi x}{l}\,dx = \frac{l}{m\pi}\left[\sin\frac{m\pi x}{l}\right]_{-l}^{l} = 0.$

$\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = 0 \quad m \neq n$
(as shown before)

$m = n$ gives

$$\int_{-l}^{l}\cos^2\frac{m\pi x}{l}\,dx = \frac{1}{2}\int_{-l}^{l}\left(1+\cos\frac{2m\pi x}{l}\right)dx$$

$$= \frac{1}{2}[x]_{-l}^{l} = \frac{1}{2}2l = l$$

Also $\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx = \int_{-l}^{0}\cos\frac{m\pi x}{l}\,dx$

$$= \frac{l}{m\pi}\left[\sin\frac{m\pi x}{l}\right]_{-l}^{0} = 0.$$

Combining, we get $\tilde{A}_m = 0$ or $\tilde{A}_n = 0$.

Now multiply by $\sin\left(\frac{m\pi x}{l}\right)$ throughout and integrate

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = A_0\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)dx$$

$$+\sum_{n=1}^{\infty}\tilde{A}_n\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$+\tilde{B}_n\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

Again, $\displaystyle\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)dx = -\frac{l}{m\pi}\left[\cos\left(\frac{m\pi x}{l}\right)\right]_{-l}^{l}$

$$= -\frac{l}{m\pi}\left[\cos m\pi - \cos(-m\pi)\right] = 0$$

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = 0$$

$$\int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0 \quad m\neq n$$

$$= l \quad m=n$$

and $\displaystyle\int_{-l}^{l} f(x)\sin\left(m\frac{\pi x}{l}\right)dx = \int_{-l}^{0}\sin\frac{m\pi x}{l}\,dx$

$$= -\frac{l}{m\pi}\left[\cos\frac{m\pi x}{l}\right]_{-l}^{0}$$

$$= -\frac{l}{m\pi}\left[1-(-1)^m\right]$$

This gives $\tilde{B}_n = -\dfrac{l}{m\pi}\left[1-(-1)^m\right]\cdot\dfrac{1}{l}$.