6) $\displaystyle\int_{(1,0)}^{(3,4)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

Step 1 $P = \dfrac{x}{\sqrt{x^2+y^2}}$; $Q = \dfrac{y}{\sqrt{x^2+y^2}}$

$$\left.\begin{aligned}\frac{\partial P}{\partial y} &= -\frac{1}{2}\frac{x \cdot 2y}{(x^2+y^2)^{-3/2}}\\ \frac{\partial Q}{\partial x} &= -\frac{1}{\cancel{2}}\frac{y \cdot \cancel{2}x}{(x^2+y^2)^{-3/2}}\end{aligned}\right\} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Hence potential function ϕ exists

Step 2 $\dfrac{\partial \phi}{\partial x} = P = \dfrac{x}{\sqrt{x^2+y^2}}$ — (1)

$\dfrac{\partial \phi}{\partial y} = Q = \dfrac{y}{\sqrt{x^2+y^2}}$ — (2)

Integrate (1) w.r.t. x

$$\phi(x,y) = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+y^2}}\,dx$$

$$= \frac{1}{\cancel{2}}\,\cancel{2}\,(x^2+y^2)^{1/2} + g(y)$$

Diff. w.r.t. y

$$\frac{\partial \phi}{\partial y} = \frac{1}{\cancel{2}}(\cancel{2}y)(x^2+y^2)^{-1/2} + g'(y)$$

Compare with (2)

$g'(y) = 0 \Rightarrow g(y) = C$ constant

C can be chosen to be zero

$$\phi(x,y) = (x^2+y^2)^{1/2}$$

Step 3 Integral is independent of path so

$$I = \phi(3,4) - \phi(1,0)$$
$$= 5 - 1 = 4.$$

15) $E(x,y) = (x^3+y)\,\underline{i} + (x+y^3)\,\underline{j}$

$P = x^3+y$; $Q = x+y^3$

$\dfrac{\partial P}{\partial y} = 1$, $\dfrac{\partial Q}{\partial x} = 1$

Hence E is conservative and potential function exists.

Step 2 $\dfrac{\partial \phi}{\partial x} = P = x^3+y$ — (1)

$\dfrac{\partial \phi}{\partial y} = Q = x+y^3$ — (2)

Integrate (1) w.r.t. x

$$\phi(x,y) = \frac{1}{4}x^4 + xy + g(y)$$

Diff w.r.t. y

$$\frac{\partial \phi}{\partial y} = x + g'(y)$$

Compare with (2)

$$g'(y) = y^3$$

This gives $g(y) = \frac{1}{4}y^4$ (ignoring constant)

Hence

$$\phi(x,y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + xy.$$

18) $P = 2x + e^{-y}$, $Q = 4y - xe^{-y}$

$\dfrac{\partial P}{\partial y} = -e^{-y}$, $\dfrac{\partial Q}{\partial x} = -e^{-y}$

Hence we can find potential function ϕ. Given integral

$$= \phi(-2,0) - \phi(2,0).$$