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On soliton and other exact solutions of the combined KdV and modified and generalized KdV equations(*)

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Summary. — In this paper we provide a rationale for using the travelling wave assumption to obtain solutions of combined KdV and modified KdV equations by using one-parameter Lie group of transformations. Some new solutions of the modified KdV equation are obtained and the procedure is applied to find the travelling wave solutions of the combined and generalized KdV equations.

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1. – Introduction

The Korteweg-de Vries (KdV) equation which models shallow-water phenomena (involving a third-order dispersion) has been analyzed extensively using the invariance properties that occur from the Lie point symmetry generator that admit it. In particular, travelling wave solutions arise from the combination of translations in space and time. Also, Galilean invariants and scale-invariant solutions are dependent on first and second Painleve transcendent (see Olver [1]). Further, the modified KdV (mKdV) has attracted interest in a similar way and its Lie point symmetry generators are known (see [2]). Recently, the combined KdV (cKdV) and mKdV

$$u_t + \alpha(1+\beta u)uu_x + \gamma u_{xxx} = 0, \quad \alpha, \gamma > 0 \tag{1.1}$$

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has been studied using various methods with a special reference to soliton-type solutions. For example, simple soliton solutions for a particular form of (1.1) used in plasma and fluid physics is obtained in [3]. Here, the particular form uses the fact that the equation admits a scaling symmetry which is nonexistent for the general cKdV (1.1). In [4], a simple N-soliton solution of (1.1) is presented by transforming it to the mKdV and then performing a travelling wave substitution. Solitary wave solutions for KdV-type equations are also given in [5]. Equation (1.1) includes the Gardner equation with $\alpha = -6$ and $\beta = 2\delta$.

In this paper, we present the travelling wave substitution on (1.1) emphasizing that this is a consequence of the equation admitting Lie point symmetry generators $\partial/\partial t$ (time translation) and $\partial/\partial x$ (translation in x). Significantly, we then show that other invariant solutions are possible by exploring whether the equation admits other point symmetries which are not obvious.

2. – Invariant solutions

It can be shown that (1.1) generates a Lie algebra of point symmetry generators (see [1] or [2] for the method of determining these) given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = -3t\frac{\partial}{\partial t} + \left(\frac{\alpha}{2\beta} - x\right)\frac{\partial}{\partial x} + \frac{1}{2}\left(\frac{1}{\beta} + 2u\right)\frac{\partial}{\partial u}. \tag{2.1}$$

Travelling wave solutions. The combination $X_1 + cX_2$ gives rise to the change of variable $y = x - ct$ and $w = u$ ($w = w(y)$), corresponding to the travelling wave solution of (1.1). In short, it can be shown that this substitution leads to an ordinary differential equation (o.d.e.) whose solution is

$$x - ct = \int^{u} \frac{1}{\sqrt{6cw^2 - 2\alpha w^3 - \alpha\beta w^4 + k_2 w + k_1}}\,\mathrm{d}w. \tag{2.2}$$

For example, with $c = 1$, $\alpha = 3$, $\beta = 2$, $k_2 = 1$, $k_1 = 0$, Mathematica yields

$$x - ct = \frac{\sqrt{2w(w-1)}(1+w)\arctan h\sqrt{2w/(w-1)}}{\sqrt{(w-1)(w+1)^2}}.$$

Other invariant solutions. As eq. (1.1) is invariant under the one-parameter Lie group of transformations with generator X_3, one needs to pursue the possible exact solutions that arise from it in the manner above requiring more details and more involved calculation. Indeed, the solution, if any, will be interesting and should not be ignored especially when one makes comparisons with numerical schemes applied to (1.1).

Firstly, the invariants of X_3 are

$$y = \frac{x}{t^{-1/3}} + \frac{\alpha}{4\beta}t^{2/3}, \qquad w = \left(\frac{1}{2\beta} + u\right)t^{1/3}, \tag{2.3}$$

so that (1.1) can be reduced to an o.d.e. involving $w = w(y)$, one integration of which leads to the second-order o.d.e.:

$$\gamma w'' - \frac{1}{3} y w + \frac{1}{3}\alpha\beta w^3 = 0\,. \tag{2.4}$$

Equation (2.4) is a second Painleve transcendent which often comes up in the reduction of certain p.d.e.'s and is by no means trivial. It generates no Lie point symmetries and, hence, from a reduction perspective, the interesting Langrangian (for $\gamma = 1$) $L = (1/2){w'}^2 + (1/6)yw^2 - (1/12)\alpha\beta w^4$ provides no relevant information. For determining invariances of the action integral and double reduction using Noether symmetries we refer the reader to [1] and [6]. Ince [7] describes how one may reduce second Painleve transcendent to a first Painleve transcendent whose analytical solution is obtainable as described in [7]. When y and w are replaced, we have another exact solution of (1.1) which, as mentioned before, can be critical in studying the numerical description of the solution.

Note. In the special case presented in [3], viz.,

$$u_t + 6uu_x + 6u^2 u_x + u_{xxx} = 0 \tag{2.5}$$

an arbitrary change of variable is presented and a travelling wave solution (soliton solution) is obtained equivalent to the general one above. However, a further analysis shows that another one-parameter Lie group of transformation with generator X_3 gives rise to a change of variables $y = xt^{-1/3} + (3/2)t^{2/3}$ and $u = wt^{-1/3} - 1/2$, $(w = w(y))$ so that a second Painleve transcendent

$$w'' + 2w^3 - \frac{1}{3} y w = 0 \tag{2.6}$$

is obtained.

3. – The generalized KdV equation

In this section, we show that one can apply the above procedure to obtain exact solutions of the generalizations of the generalized KdV equation

$$u_t + 6u^n u_x + u_{xxx} = 0, \quad n \neq 0, 1, 2 \tag{3.1}$$

which admits the translation symmetries in "t" and "x" leading to travelling wave solutions discussed below in the combined equation. Also, it admits the dilation symmetry generator

$$3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{n}u\frac{\partial}{\partial u}$$

with invariants giving new variables

$$y = \frac{x^3}{t}, \qquad w = t^{2/(3n)}u\,,$$

so that (3.1) becomes the third-order o.d.e.:

$$27y^2w''' + 54yw'' + 3y^{2/3}w^n w' - yw' + 6w' + w = 0\,, \tag{3.2}$$

which, unlike most of the similarity reductions of the various forms of the KdV, cannot be once integrated to a second-order o.d.e. Equation (3.2) is not easily solvable but provides a non-trivial similarity solution and can be useful in the numerical treatment of the generalized KdV.

In a similar way, symmetry analysis of the combined KdV and the generalized KdV equation

$$u_t + \alpha(1+\beta u^{n-1})uu_x + \gamma u_{xxx} = 0, \quad \alpha,\gamma > 0,\ n \neq -2,-1,1,2 \tag{3.3}$$

having symmetries $\partial/\partial t$ and $\partial/\partial x$ can be performed for travelling wave solutions to obtain

$$x - ct = \int^{u} \frac{1}{\sqrt{cw^2 - \frac{1}{3}\alpha w^3 - \frac{2}{(n+1)(n+2)}\alpha\beta w^4 + k_2 w + k_1}}\,\mathrm{d}w. \tag{3.4}$$

4. – Conclusion

We have shown how similarity studies of the various KdV equations produce interesting solutions by which new are obtainable. Whilst these solutions are important, they are significant when numerical analyses are carried out. Also, the method gives an alternative procedure to obtaining the physically important soliton (travelling wave)-type solutions.

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