Math 301

12.5

(8) $y'' + y' + \lambda y = 0$, $y(0) = 0$, $y(2) = 0$

Self-adjoint from I.F. $= e^{\int 1 dx} = e^x$

Multiply throughout by e^x

$e^x y'' + e^x y' + \lambda e^x y = 0 \Rightarrow \frac{d}{dx}[e^x y'] + \lambda e^x y = 0$

Thus weight function $p(x) = e^x$

orthogonality relation : $\int_0^2 e^x y_m(x) y_n(x)\, dx = 0$, $m \neq n$.

Eigenvalues and eigenfunctions

Auxiliary equation : $m^2 + m + \lambda = 0$

$$m = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

In this case we have to consider cases $1 - 4\lambda = 0$, > 0 and < 0.

Case(a) $1 - 4\lambda = 0$, this gives $m = -\frac{1}{2}, -\frac{1}{2}$

$y = c_1 e^{-\frac{1}{2}x} + x c_2 e^{-\frac{1}{2}x}$

$y(0) = 0 \Rightarrow c_1 = 0$, so $y = c_2 x e^{-\frac{1}{2}x}$

$y(2) = 0 \Rightarrow 2 c_2 e^{-1} = 0 \Rightarrow c_2 = 0$ (trivial solution)

Case(b) $1 - 4\lambda > 0$, $m = -\frac{1}{2} \pm \frac{k}{2} = \frac{1}{2}(-1 \pm k)$, k real

say $1 - 4\lambda = k^2$

$y = c_1 e^{-\frac{1}{2}(1-k)x} + c_2 e^{-\frac{1}{2}(k+1)x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y(2) = 0 \Rightarrow c_1 e^{-\frac{1}{2}(1-k)} + c_2 e^{-\frac{1}{2}(k+1)} = 0 \Rightarrow c_1 \left(e^{-\frac{1}{2}(1-k)} - e^{-\frac{1}{2}(k+1)} \right) = 0$

$\Rightarrow e^{-\frac{1}{2}} \left(e^{+\frac{1}{2}k} - e^{-\frac{1}{2}k} \right) = 0 \Rightarrow k = 0$ same as case(a)

Case(c) $1 - 4\lambda < 0$, Put $1 - 4\lambda = -k^2$, k real

$m = -\frac{1}{2} \pm \frac{1}{2} i k$ $\qquad \alpha = -\frac{1}{2}$, $\beta = \frac{k}{2}$

$y(x) = c_1 e^{-\frac{1}{2}x} \cos \frac{k}{2} x + c_2 e^{-\frac{1}{2}x} \sin \frac{k}{2} x$

$y(0) = 0 \Rightarrow c_1 = 0$, so $y = c_2 e^{-\frac{1}{2}x} \sin \frac{k}{2} x$

$y(2) = 0 \Rightarrow c_2 e^{-1} \sin k = 0 \Rightarrow k = n\pi$, $n = 1, 2, 3, \ldots$

(as $c_2 \neq 0$, $e^{-1} \neq 0$)

Thus $\lambda = \frac{1 + n^2\pi^2}{4}$, $y_n(x) = c_n e^{-\frac{1}{2}x} \sin \frac{n\pi}{2} x$.