THE ALTERNATING GROUP EXPLICIT (AGE) ITERATIVE METHOD FOR SOLVING A LADYZHENSKAYA MODEL FOR STATIONARY INCOMPRESSIBLE VISCOUS FLOW *

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Abstract. In this paper, the alternating group explicit (AGE) iterative method is applied to a nonlinear 4th order PDE describing the flow of an incompressible fluid. This equation is a Ladyzhenskaya equation. The AGE method is shown to be extremely powerful and flexible and affords its users many advantages. Computational results are obtained to demonstrate the applicability of the method on some problems with known solutions. This paper demonstrates that the (AGE) method can be implemented to approximate efficiently solutions to the Navier-Stokes equations and the Ladyzhenskaya equations. Problems with a known solution are considered to test the method and to compare the computed results with the exact values. Streamfunction contours and some plots are displayed showing the main features of the solution.

keywords: Alternating Group Explicit (AGE) method, Ladyzhenskaya equations, Navier-Stokes equations.

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1. INTRODUCTION. Understanding turbulent flow is central to many important problems including environmental and energy related applications (global change, mixing of fuel and oxidizer in engines and drag reduction), aerodynamics (maneuvering flight of jet aircraft) and biophysical applications (blood flow in the heart). However, in many situations it is still not clear which models are most appropriate, especially in the case of turbulent flows.

The Navier-Stokes equations are generally accepted as providing an accurate model for the incompressible motion of viscous fluids in practical situations. This research will consider one model introduced by Ladyzhenskaya [11, 12, 13]. The study of this model may be justified through a variety of physical and mathematical arguments. The following paragraphs address the reasons for choosing the Ladyzhenskaya model. (These reasons appear in [2, 3] and have been summarized in the following paragraphs).

The first reason for the study of the Ladyzhenskaya model is from a modeling stand point. The Stokes hypotheses which define an ordinary fluid (water or air, for example) lead to a specific mathematical form of the nonlinear relation between the stress and the velocity fields. If one requires that the relation between the stress and the velocity be linear, then one arrives at the Navier-Stokes equations. However, if one retains the Stokes hypotheses defining a fluid and then retains some of the nonlinear terms in the general constitutive relation which a Stoksian fluid must satisfy, then one arrives at the Ladyzhenskaya model considered here, see [12, 13]. Thus, from a modeling stand point, the Navier-Stokes equations are a special case of the Ladyzhenskaya equations. This leads to the obvious conclusion that any flow which can be accurately described by solutions of the Navier-Stokes equations can be at least as accurately described by solutions of the Ladyzhenskaya equations.

The second reason for the study of the Ladyzhenskaya model comes from the field of turbulence modeling. The velocity and pressure (the variables of the problem) may be decomposed into the sum of averaged and fluctuating quantities. The averaged form of the Navier-Stokes equations can not themselves determine the averaged quantities; one must also provide a relation between the fluctuating and averaged quantities, which determine how energy is transferred from the small scales to the larger scales in the flow. The various methods for relating the mean velocity fields and the Reynolds stresses arising from the fluctuating velocity field are known collectively as turbulence closure models. One class of such models is known as algebraic or zero models. It turns out again for certain values of the parameter p, the Ladyzhenskaya equations considered here are identical to this of a popular algebraic turbulence model. At some specific values of the parameters, the Ladyzhenskaya equations become

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the Smagorinsky model [17]. The Smagorinsky model, which is widely recognized, is the most popular model in Large Eddy Simulation (LES). Thus, from a practical engineering point of view, the study of Ladyzhenskaya equations and of proprieties of their solution is of substantial interest.

The (AGE) method is an iterative method which employs the fractional splitting strategy which is applied alternately at each intermediate step on tridiagonal system of difference schemes. Its rate of convergence is governed by the acceleration parameter r. The (AGE) iterative method is applied to a variety of problems involving parabolic and hyperbolic partial differential equations (see [4, 7, 5, 6]). In [15], Sahimi and Evans reformulated the (AGE) method to solve the Navier-Stokes equations in the streamfunction-vorticity form. In this paper we applay the (AGE) iterative method to the Ladyzhenskaya equations in the streamfunction-vorticity form.

2. GOVERNING EQUATIONS. In [11, 12, 13], a model for the motion of ideal incompressible viscous flow has been proposed by Ladyzhenskaya in terms of the velocity and the pressure. Further studies are made in [2, 3, 1, 9, 14]. The Ladyzhenskaya model in the streamfunction ψ form with its finite element analysis are studied in [10]. The streamfunction equation of the Ladyzhenskaya model is a fourth-order partial differential equation. Using an initial approximation ψ^* of the streamfunction ψ and reducing the fourth-order PDE into a coupled system of second-order PDEs, by introducing the vorticity $\omega = -\triangle\,\psi$, yield the studied equations in this paper. In this work we study computational aspects of the Ladyzhenskaya model in terms of the streamfunction ψ and the vorticity ω. The model we work with is as follows:

Consider the following coupled system of partial differential equations in the dependent variable ψ and ω:

$$\triangle\psi = -\omega, \tag{2.1}$$

$$\triangle(\tilde{A}(\psi^*)\omega) + Re(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}) = -g, \tag{2.2}$$

where x and y are independent variables with a set of a boundary conditions prescribed on a square region of the xy-plane. Here $\triangle$ is the usual Laplacian operator defined by,

$$\triangle\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2},$$

where in (2.2) $\tilde{A}(\psi^*)$ is defined by

$$\tilde{A}(\psi^*) = 1 + Re\ \epsilon_1 \mid \overrightarrow{\triangle}\psi^* \mid^{q-2},$$

with Re, ϵ_1 and $q-2>0$ and

$$\overrightarrow{\triangle}\psi = \overrightarrow{grad}\ (\ \overrightarrow{grad}\ \psi\ ) = [\psi_{xx}, \psi_{xy}, \psi_{yx}, \psi_{yy}]^T,$$

and

$$\mid \overrightarrow{\triangle}\psi \mid = (\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2)^{\frac{1}{2}}.$$

Note that if $Re = 0$, then equations (2.1) and (2.2) define a biharmonic equation given by,

$$\triangle^2\psi = \frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = -g.$$

If $Re \neq 0$ and $\epsilon_1 = 0$, then equations (2.1) and (2.2) become the Navier-Stokes equations;

$$\triangle\psi = -\omega,$$

$$\triangle\omega + Re(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}) = -g;$$

which describe the basic two dimensional, steady-state, viscous, incompressible flow problem.

3. FINITE DIFFERENCE DISCRETISATION. Let Ω be a square region of the solution domain defined by,

$$\Omega = \{(x, y) : 0 \leq x \leq L, 0 \leq y \leq L\}. \tag{3.1}$$

A uniformly spaced network whose mesh points are $x_i = ih$, $y_j = jh$, with $h = L/(m+1)$ for $i, j = 0, 1, \cdots, m, m+1$ is now superimposed on Ω.

It is observed that if ω is known, then (2.1) is a linear elliptic equation in ψ, while if ψ is known, then (2.2) is a linear elliptic equation in ω. Using central difference approximations, equations (2.1) and (2.2) can now be descritised at the grid point (x_i, y_j) by the following finite difference equations,

$$-\psi_{i-1,j}^{(k+1)} - \psi_{i,j-1}^{(k+1)} - \psi_{i,j+1}^{(k+1)} - \psi_{i+1,j}^{(k+1)} + 4\psi_{i,j}^{(k+1)} = h^2\omega_{i,j}^{(k)}, \tag{3.2}$$

$$\begin{aligned}
&-[\tilde{A}_{i-1,j}^{(k+1)} - \alpha(\psi_{i,j-1}^{(k+1)} - \psi_{i,j+1}^{(k+1)})]\omega_{i-1,j}^{(k+1)} - [\tilde{A}_{i,j-1}^{(k+1)} + \alpha(\psi_{i-1,j}^{(k+1)} - \psi_{i+1,j}^{(k+1)})]\omega_{i,j-1}^{(k+1)} \\
&-[\tilde{A}_{i,j+1}^{(k+1)} - \alpha(\psi_{i-1,j}^{(k+1)} - \psi_{i+1,j}^{(k+1)})]\omega_{i,j+1}^{(k+1)} - [\tilde{A}_{i+1,j}^{(k+1)} + \alpha(\psi_{i,j-1}^{(k+1)} - \psi_{i,j+1}^{(k+1)})]\omega_{i+1,j}^{(k+1)} \\
&+4\tilde{A}_{i,j}^{(k+1)}\omega_{i,j}^{(k+1)} = h^2 g_{i,j},
\end{aligned} \tag{3.3}$$

where $\alpha = Re/4$ and

$$\tilde{A}_{i,j}^{(k+1)} = 1 + Re\ \epsilon_1 \mid \overrightarrow{\triangle}\psi_{i,j}^{(k+1)} \mid^{q-2} \quad \text{and } i, j = 1, 2, \cdots, m\ , \tag{3.4}$$

and

$$\mid \overrightarrow{\triangle}\psi_{i,j}^{(k+1)} \mid = \left[\left([\psi_{xx}]_{ij}^{(k+1)} \right)^2 + 2 \left([\psi_{xy}]_{ij}^{(k+1)} \right)^2 + \left([\psi_{yy}]_{ij}^{(k+1)} \right)^2 \right]^{\frac{1}{2}},$$

and

$$\begin{aligned}
[\psi_{xx}]_{ij}^{(k+1)} &= \frac{1}{h^2}[\psi_{i+1,j}^{(k+1)} - 2\psi_{i,j}^{(k+1)} + \psi_{i-1,j}^{(k+1)}] \\
[\psi_{xy}]_{ij}^{(k+1)} &= \frac{1}{4h^2}[\psi_{i+1,j+1}^{(k+1)} - \psi_{i-1,j+1}^{(k+1)} - \psi_{i+1,j-1}^{(k+1)} + \psi_{i-1,j-1}^{(k+1)}] \\
[\psi_{yy}]_{ij}^{(k+1)} &= \frac{1}{h^2}[\psi_{i,j+1}^{(k+1)} - 2\psi_{i,j}^{(k+1)} + \psi_{i,j-1}^{(k+1)}].
\end{aligned}$$

Equation (3.2) and (3.3) suggest that we start with an initial guess $\omega^{(0)}$ and use equation (3.2) to approximate ψ and call this $\psi^{(1)}$ and use this to solve for ω using equation (3.3) and call this as $\omega^{(1)}$. Continue this computations until you reach a specific convergence criterion. We will call this process an outer iteration.

We will study in detail the finite-difference analogue of the vorticity equation (3.3) to derive the AGE equations for its solution. Then, the AGE equations for the streamfunction equation (3.2) will follow, since equation (3.2) is similar to equation (3.3) but with different coefficients.

4. THE AGE METHOD. If we use the boundary conditions $\psi = 0$ and $\partial^2\psi/\partial\hat{n}^2 = 0$ where $\hat{n}$ denotes the normal to the boundary $\partial\Omega$ of Ω, then our problem amounts to solving successively (2.1) and (2.2) with $\psi = 0$ and $\omega = 0$ along $\partial\Omega$. First, let us rewrite Equation (3.3) in matrix form as:

$$A\underline{\omega}_{(r)}^{k+1} = \underline{f},$$

$$\text{where,} \quad \underline{\omega}_{(r)} = (\underline{\omega}_1, \underline{\omega}_2, \cdots, \underline{\omega}_m)^T \quad \text{with} \quad \omega_j = (\omega_{1j}, \omega_{2j}, \cdots, \omega_{mj})^T,$$

$j = 1, 2, \cdots, m$ i.e. the m^2 internal grid points are ordered row-wise parallel to the x-axis on the square mesh, $\underline{f} = (\underline{f}_1, \underline{f}_2, \cdots, \underline{f}_m)^T$ with,

$$f_j = h^2(g_{1j}, g_{2j}, \cdots, g_{mj})^T \quad \text{and} \quad g_{ij} = g(x_i, y_j) \quad \text{for} \quad i, j = 1, 2, \cdots, m, \tag{4.1}$$

and,

$$A = \begin{bmatrix} A_1 & B_1 & & & & & \\ C_2 & A_2 & B_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & & & \\ & & & & \ddots & \ddots & \ddots \\ & & & & C_{m-1} & A_{m-1} & B_{m-1} \\ & & & & & C_m & A_m \end{bmatrix}_{(m^2\times m^2)},$$

$$A_j = \begin{bmatrix} 4 & \beta_{1,j+1} & & & & \\ \hat{\beta}_{2,j+1} & 4 & \beta_{2,j+1} & & & \\ \ddots & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \hat{\beta}_{m-1,j+1} & 4 & \beta_{m-1,j+1} \\ & & & \hat{\beta}_{m,j+1} & 4 \end{bmatrix}_{m\times m}, \quad j = 1, 2, \cdots, m;$$

$$B_j = \mathrm{diag}(\mu_{1j}, \mu_{2j}, \cdots, \mu_{m-1,j}, \mu_{mj}), \quad j = 1, 2, \cdots, m-1,$$

and

$$C_j = \mathrm{diag}(\hat{\mu}_{1j}, \hat{\mu}_{2j}, \cdots, \hat{\mu}_{m-1,j}, \hat{\mu}_{mj}), \quad j = 2, 3, \cdots, m,$$

where

$$\beta_{ij} = -[\tilde{A}^{(k+1)}_{i+1,j-1} - \alpha(\psi^{(k+1)}_{ij} - \psi^{(k+1)}_{i,j-2})], i= 1, 2, \cdots, m-1; j = 2, 3, \cdots, m+1; \tag{4.2}$$

$$\hat{\beta}_{ij} = -[\tilde{A}^{(k+1)}_{i-1,j-1} + \alpha(\psi^{(k+1)}_{ij} - \psi^{(k+1)}_{i,j-2})], \quad i = 2, 3, \cdots, m; j = 2, 3, \cdots, m+1; \tag{4.3}$$

$$\mu_{ij} = -[\tilde{A}^{(k+1)}_{i,j+1} + \alpha(\psi^{(k+1)}_{i+1,j} - \psi^{(k+1)}_{i-1,j})], \quad i = 1, 2, \cdots, m; j = 2, 3, \cdots, m-1; \tag{4.4}$$

and

$$\hat{\mu}_{ij} = -[\tilde{A}^{(k+1)}_{i,j-1} - \alpha(\psi^{(k+1)}_{i+1,j} - \psi^{(k+1)}_{i-1,j}], \quad i = 1, 2, \cdots, m; \quad j = 2, 3, \cdots, m. \tag{4.5}$$

If we split A into the sum of its constituent matrices G_1, G_2, G_3, G_4 as,

$$A = G_1 + G_2 + G_3 + G_4,$$

then following [15] we have,

$$G_1 + G_2 = \mathrm{diag}(\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_m)_{(m^2\times m^2)},$$

and

$$G_3 + G_4 = \begin{bmatrix} D & B_1 & & & & & \\ C_2 & D & B_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & & & & \\ & & & & C_{m-1} & D & B_{m-1} \\ & & & & & C_m & D \end{bmatrix}_{(m^2\times m^2)},$$

where,

$$\hat{A}_j = \begin{bmatrix} 2 & \beta_{1,j+1} & & & & \\ \hat{\beta}_{2,j+1} & 2 & \beta_{2,j+1} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \hat{\beta}_{m-1,j+1} & 2 & \beta_{m-1,j+1} \\ & & & \hat{\beta}_{m,j+1} & 2 \end{bmatrix}_{m\times m}, \quad j = 1, 2, \cdots, m;$$

and

$$D = \text{diag}(2, 2, \cdots, 2).$$

The AGE Douglas fractional formulae then are written in the following form,

$$(G_1 + rI)\underline{\omega}_{(r)}^{(p+1/4)} = ((rI + G_1) - 2A)\underline{\omega}_{(r)}^{(p)} + 2\underline{f}, \tag{4.6}$$

$$(G_2 + rI)\underline{\omega}_{(r)}^{(p+1/2)} = G_2\underline{\omega}_{(r)}^{(p)} + r\omega_{(r)}^{(p+1/4)}, \tag{4.7}$$

$$(G_3 + rI)\underline{\omega}_{(r)}^{(p+3/4)} = G_3\underline{\omega}_{(r)}^{(p)} + r\omega_{(r)}^{(p+1/2)}, \tag{4.8}$$

$$(G_4 + rI)\underline{\omega}_{(r)}^{(p+1)} = G_4\underline{\omega}_{(r)}^{(p)} + r\omega_{(r)}^{(p+3/4)}. \tag{4.9}$$

In equations (4.6 - 4.9), p represents the index for the inner iteration procedure and r is the acceleration parameter. Without loss of generality, we assume that m is odd.

From equations (4.6 - 4.9), we will write $\omega_{ij}^{p+1/4}, \omega_{ij}^{p+1/2}, \omega_{ij}^{p+3/4}, \omega_{ij}^{p+1}$ in explicit form. We start by multiplying equation (4.6) by the inverse of the matrix $(rI + G_1)$. But the matrix $(rI + G_1)$ is a block-diagonal matrix of 2×2 or 1×1 matrices. Fortunately, we have a closed form for the inverse of $(rI + G_1)$. After some mathematical manipulations, we write $\omega_{ij}^{p+1/4}$ in an explicit form in terms of $\omega_{ij}^{(p-1)}$, β_{ij}, $\hat{\beta}_{ij}$, μ_{ij}, $\hat{\mu}_{ij}$, f_{ij}. Then, we repeat the same process to write $\omega_{ij}^{(p+1/2)}$ by using equation (4.7). To write $\omega_{ij}^{(p+3/4)}$ and $\omega_{ij}^{(p+1)}$ in explicit form, we start by reordering the mesh points column-wise parallel to the y-axis then we apply the same process. Now, let us start at $(p + 1)th$ iterate.

(i) At the $(p + 1/4)^{\mathbf{th}}$ iterate

From (4.6) we have

$$\underline{\omega}_{(r)}^{(p+1/4)} = (G_1 + rI)^{-1}[((rI + G_1) - 2A)\underline{\omega}_{(r)}^{(p)} + 2\underline{f}].$$

We find that,

$$rI + G_1 = \text{diag}(\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_{m-1}, \hat{C}_m)_{(m^2\times m^2)}, \tag{4.10}$$

where

$$\hat{C}_j = \text{diag}(r_1, \hat{G}_{2,j}, \hat{G}_{4,j}, \cdots, \hat{G}_{m-1,j})_{(m\times m)} \quad \text{for} \quad j = 1, 3, \cdots, m(\text{odd}),$$

and

$$\hat{C}_j = \text{diag}(\hat{G}_{1,j}, \hat{G}_{3,j}, \cdots, \hat{G}_{m-2,j}, r_1)_{m\times m}, \quad \text{for} \quad j = 2, 4, \cdots, m(\text{even}),$$

with

$$r_1 = r + 1,$$

and

$$\hat{G}_{i,j} = \begin{bmatrix} r_1 & \beta_{i,j+1} \\ \hat{\beta}_{i+1,j+1} & r_1 \end{bmatrix}, \quad i = 1, 2, \cdots, m-1\ .$$

Since $rI + G_1$ is block diagonal, (4.10) gives,

$$(rI + G_1)^{-1} = \text{diag}(\hat{C}_1^{-1}, \hat{C}_2^{-1}, \cdots, \hat{C}_{m-1}^{-1}, \hat{C}_m^{-1})_{(m^2\times m^2)}.$$

Defining

$$\begin{aligned} D_i &= \hat{C}_i - 2A_i, && i = 1, 2, \cdots, m; \\ E_i &= -2C_i, && i = 2, 3, \cdots, m; \\ F_i &= -2B_i, && i = 1, 2, \cdots, m-1; \end{aligned}$$

we obtain the following set of equations at the $(p + 1/4)^{\text{th}}$ iterate,

$$\underline{\omega}_{1(r)}^{(p+1/4)} = \hat{C}_1^{-1}(D_1\underline{\omega}_{1(r)}^{(p)} + F_1\underline{\omega}_{2(r)}^{(p)} + 2\underline{f}_1), \tag{4.11}$$

$$\underline{\omega}_{j(r)}^{(p+1/4)} = \hat{C}_j^{-1}(E_j\underline{\omega}_{j-1(r)}^{(p)} + D_j\underline{\omega}_{j(r)}^{(p)} + F_j\underline{\omega}_{j+1(r)}^{(p)} + 2\underline{f}_j), \qquad j = 2, 3, \cdots, m-2, m-1, \tag{4.12}$$

and

$$\underline{\omega}_{m(r)}^{(p+1/4)} = \hat{C}_m^{-1}(E_m\underline{\omega}_{m-1(r)}^{(p)} + D_m\underline{\omega}_{m(r)}^{(p)} + 2\underline{f}_m), \tag{4.13}$$

and

$$\hat{C}_j^{-1} = \text{diag}(\frac{1}{r_1}, (\hat{G}_{2,j})^{-1}, ((\hat{G}_{4,j})^{-1}, \cdots, ((\hat{G}_{m-1,j})^{-1})_{(m\times m)}, \quad \text{for} j = 1, 3, \cdots, m;$$

$$\hat{C}_j^{-1} = \text{diag}((\hat{G}_{1,j})^{-1}, ((\hat{G}_{3,j})^{-1}, \cdots, ((\hat{G}_{m-2,j})^{-1}, \frac{1}{r_1})_{(m\times m)}, \quad \text{for} j = 2, 4, \cdots, m-1;$$

with

$$(\hat{G}_{i,j})^{-1} = \frac{1}{\triangle_{i,j}} \begin{bmatrix} r_1 & -\beta_{i,j+1} \\ -\hat{\beta}_{i+1,j+1} & r_1 \end{bmatrix},$$

and

$$\triangle_{i,j} = r_1^2 - \beta_{i,j+1}\hat{\beta}_{i+1,j+1}.$$

Writing equation (4.11) component-wise gives

$$\omega_{11}^{(p+1/4)} = 2[\frac{r_2}{2}\omega_{11}^{(p)} - \beta_{12}\omega_{21}^{(p)} - \mu_{11}\omega_{21}^{(p)} + f_{11}]/r_1, \tag{4.14}$$

$$\begin{aligned} \omega_{i,1}^{(p+1/4)} &= 2(a_i\omega_{i-1,1}^{(p)} + b_i\omega_{i,1}^{(p)} + c_i\omega_{i+1,1}^{(p)} + d_i\omega_{i+2,1}^{(p)} + e_i\omega_{i,2}^{(p)} \\ &\quad + f_i\omega_{i+1,2}^{(p)} + g_i)/\triangle_{i,1}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \omega_{i+1,1}^{(p+1/4)} &= 2(\overline{a}_i\omega_{i-1,1}^{(p)} + \overline{b}_i\omega_{i,1}^{(p)} + b_i\omega_{i+1,1}^{(p)} + \overline{d}_i\omega_{i+2,1}^{(p)} + \overline{e}_i\omega_{i,2}^{(p)} \\ &\quad + \overline{f}_i\omega_{i+1,2}^{(p)} + \overline{g}_i)/\triangle_{i,1}, \quad \text{for } i = 2, 4, \cdots, m-3, m-1, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} & a_i = -r_1\hat{\beta}_{i,2}, b_i = (r_1r_2 + \beta_{i,2}\hat{\beta}_{i+1/2})/2, c_i = -r_3\beta_{i,2}, \\ & e_i = -r_1\mu_{i,1}, f_i = \beta_{i,2}\mu_{i+1,1}, g_i = r_1 f_{i,1} - \beta_{i,2}f_{i+1,1}, \\ & \overline{a}_i = \hat{\beta}_{i,2}\hat{\beta}_{i+1,2}, \overline{b}_i = -r_3\hat{\beta}_{i+1,2}, \overline{e} = \hat{\beta}_{i+1,2}\mu_{i,1}, \\ & \overline{f}_i = -r_1\mu_{i+1,1}, \overline{g}_i = r_1 f_{i+1,1} - \hat{\beta}_{i+1,2}f_{i,1}, \\ & r_2 = r_1 - 8, r_3 = r_1 - 4, \\ & d_i = \begin{cases} \beta_{i,2}\beta_{i+1,2}, & \text{if} \quad i \neq m-1; \\ 0, & \text{if} \quad i = m-1, \end{cases} \\ & \overline{d}_i = \begin{cases} -r_1\beta_{i+1,2}, & \text{if} \quad i \neq m-1; \\ 0, & \text{if} \quad i = m-1. \end{cases} \end{aligned}$$

In the same manner, writing equations (4.12) and (4.13)component-wise give

$$\left\{\begin{array}{ll}\omega_{i,j}^{(p+1/4)} & = 2(r_1 q_{i,j} - \beta_{i,j+1}\overline{q}_{ij})/\triangle_{ij}\\ \omega_{i+1,j}^{(p+1/4)} & = 2(-\hat{\beta}_{i+1,j+1}q_{ij} + r_1\overline{q}_{ij})/\triangle_{ij}\end{array}\right\}, j = 2,4,\cdots,m-1; i = 1,3,\cdots,m-2, \quad (4.17)$$

$$\omega_{m,j}^{(p+1/4)} = 2(-\mu_{mj}\omega_{m,j-1}^{(p)} - \hat{\beta}_{m,j+1}\omega_{m-1,j}^{(p)} + \frac{r_2}{2}\omega_{mj}^{(p)} - \mu_{mj}\omega_{m,j+1}^{(p)} + f_{m,j})/r_1,$$
$$\text{for } j = 2,4,\cdots,m-1, \quad (4.18)$$

where

$$q_{ij} = \begin{cases}-\mu_{i,j}\omega_{i,j-1}^{(p)} - \hat{\beta}_{i,j+1}\omega_{i-1,j}^{(p)} + \frac{r_2}{2}\omega_{i,j}^{(p)} - \frac{\beta_{i,j+1}}{2}\omega_{i+1,j}^{(p)} - \mu_{i,j}\omega_{i,j+1}^{(p)} + f_{i,j}, & \text{if } \quad i \neq 1;\\ -\mu_{i,j}\omega_{i,j-1}^{(p)} + \frac{r_2}{2}\omega_{i,j}^{(p)} - \frac{\beta_{i,j+1}}{2}\omega_{i+1,j}^{(p)} - \mu_{i,j}\omega_{i,j+1}^{(p)} + f_{i,j},, & \text{if } \quad i = 1,\end{cases}$$

$$\overline{q_{ij}} = \begin{cases}-\hat{\mu}_{i+1,j}\omega_{i+1,j-1}^{(p)} - \frac{\hat{\beta}_{i+1,j+1}}{2}\omega_{i,j}^{(p)} + \frac{r_2}{2}\omega_{i+1,j}^{(p)} - \beta_{i+1,j+1}\omega_{i+2,j}^{(p)} - \\ \qquad \mu_{i+1,j}\omega_{i+1,j+1}^{(p)} + f_{i+1,j}, & \text{if } \ i \neq m-1,\\ -\hat{\mu}_{i+1,j}\omega_{i+1,j-1}^{(p)} - \frac{\hat{\beta}_{i+1,j+1}}{2}\omega_{i,j}^{(p)} + \frac{r_2}{2}\omega_{i+1,j}^{(p)} + f_{i+1,j}, & \text{if } \ i = m-1\end{cases}$$

and

$$\left\{\begin{array}{c}\omega_{1j}^{(p+1/4)} = 2(-\hat{\mu}_{1j}\omega_{1,j-1}^{(p)} + \frac{r_2}{2}\omega_{1j}^{(p)} - \beta_{1,j+1}\omega_{2j}^{(p)} - \mu_{1j}\omega_{1,j+1}^{(p)} + f_{1j})/r_1,\\ \text{for } \ j = 3,5,\cdots,m-2\\ \omega_{1,m}^{(p+1/4)} = 2(-\hat{\mu}_{1m}\omega_{1,m-1}^{(p)} + \frac{r_2}{2}\omega_{1m}^{(p)} - \beta_{1,m+1}\omega_{2m}^{(p)} + f_{1m})/r_1\end{array}\right\}, \quad (4.19)$$

$$\left\{\begin{array}{c}\omega_{i,j}^{(p+1/4)} = 2(r_1 q_{ij} - \beta_{i,j+1}\overline{q}_{ij})/\triangle_{ij}\\ \omega_{i+1,j}^{(p+1/4)} = 2(-\hat{\beta}_{i+1,j+1}q_{ij} + r_1\overline{q}_{ij})/\triangle_{ij}\end{array}\right\}, j = 3,5,\cdots,m-2,m; i = 2,4,\cdots,m-3,m-1, \quad (4.20)$$

(ii) At the $(p+1/2)^{\text{th}}$ iterate

Equation (4.7) gives

$$\underline{\omega}_{(r)}^{(p+1/2)} = (G_2 + rI)^{-1}(G_2\underline{\omega}_{(r)}^{(p)} + r\underline{\omega}_{(r)}^{(p+1/4)}). \quad (4.21)$$

We define,

$$(rI + G_2) = \text{diag}(\hat{\hat{C}}_1, \hat{\hat{C}}_2, \cdots, \hat{\hat{C}}_{m-1}, \hat{\hat{C}}_m)_{(m^2\times m^2)},$$

where

$$\hat{\hat{C}}_j = \text{diag}(\hat{\hat{C}}_{1,j}, \hat{\hat{C}}_{3,j}, \cdots, \hat{\hat{C}}_{m-2,j}, r_1)_{(m\times m)}, \quad j = 1,3,\cdots,m(\text{odd}),$$

and

$$\hat{\hat{C}}_j = \text{diag}(r_1, \hat{G}_{2,j}, \hat{G}_{4,j}, \cdots, \hat{G}_{m-1,j})_{(m\times m)}, \quad j = 2,4,\cdots,m-1(\text{even}),$$

Denoting $\overline{C}_j \equiv \hat{\hat{C}}_j$ but the diagonal element r_1 replaced by 1, equation (4.21) becomes

$$\underline{\omega}_{j(r)}^{(p+1/2)} = (\hat{\hat{C}}_j)^{-1}(\overline{C}_j\underline{\omega}_{j(r)}^{(p)} + r\underline{\omega}_{j(r)}^{(p+1/4)}, \quad j = 1,2,\cdots,m \ ,$$

which leads to

$$\left\{\begin{array}{ll}\omega_{i,j}^{(p+1/2)} & = (r_1 s_{i,j} - \beta_{i,j+1}\overline{s}_{i,j})/\triangle_{i,j},\\ \omega_{i+1,j}^{(p+1/2)} & = (-\hat{\beta}_{i+1,j+1}s_{i,j} + r_1\overline{s}_{i,j})/\triangle_{i,j},\end{array}\right\}, j = 1,3,\cdots,m; i = 1,3,\cdots,m-2 \ , \quad (4.22)$$

$$\omega_{m,j}^{(p+1/2)} = (\omega_{m,j}^{(p)} + r\omega_{m,j}^{(p+1/4)})r_1, \qquad j = 1, 3, \cdots, m\ , \tag{4.23}$$

and

$$\omega_{1,j}^{(p+1/2)} = (\omega_{(1,j)}^{(p)} + r\omega_{(1,j)}^{(p+1/4)})/r_1, \quad j = 2, 4, \cdots, m-1\ , \tag{4.24}$$

$$\left\{ \begin{array}{l} \omega_{i,j}^{(p+1/2)} = (r_1 s_{i,j} - \beta_{i,j+1}\overline{s}_{i,j})/\triangle_{i,j}, \\ \omega_{i+1,j}^{(p+1/2)} = (-\hat{\beta}_{i+1,j+1}s_{i,j} + r_1\overline{s}_{i,j})/\triangle_{i,j}, \end{array} \right\}, j = 2, 4, \cdots, m-1; i = 2, 4, \cdots, m-1, \tag{4.25}$$

where

$$s_{i,j} = \omega_{i,j}^{(p)} + \beta_{i,j+1}\omega_{i+1,j}^{(p)} + r\omega_{i,j}^{(p+1/4)},$$

and

$$\overline{s}_{i,j} = \hat{\beta}_{i+1,j+1}\omega_{i,j}^{(p)} + \omega_{i+1,j}^{(p)} + r\omega_{i+1,j}^{(p+1/4)}.$$

(iii) At the $(p+3/4)^{\mathbf{th}}$ iterate

By ordering the mesh points column-wise parallel to the y-axis we have,

$$\underline{\omega}_{(c)} = (\underline{\omega}_1, \underline{\omega}_2, \cdots, \underline{\omega}_m)^T \quad \text{with} \quad \underline{\omega}_{(i)} = (\omega_{i1}, \omega_{i2}, \cdots, \omega_{im})^T, \quad i = 1, 2, \cdots, m,$$

and

$$(G_3 + G_4)\underline{\omega}_{(r)} = (\overline{G}_3 + \overline{G}_4)\underline{\omega}_{(c)}.$$

This reordering transforms equation (4.8) to

$$(\overline{G}_3 + rI)\underline{\omega}_{(c)}^{(p+3/4)} = \overline{G}_3\underline{\omega}_{(c)}^{(p)} + r\underline{\omega}_{(c)}^{(p+1/2)}.$$

or

$$\underline{\omega}_{(c)}^{(p+3/4)} = (\overline{G}_3 + rI)^{-1}(\overline{G}_3\underline{\omega}_{(c)}^{(p)} + r\omega_{(c)}^{(p+1/2)}. \tag{4.26}$$

Now

$$\overline{G}_3 + \overline{G}_4 = \text{diag}(\hat{B}_1, \hat{B}_2, \cdots, \hat{B}_{m-1}\hat{B}_m)_{(m^2\times m^2)},$$

where

$$\hat{B}_i = \begin{bmatrix} 2 & \mu_{i,1} & & & & \\ \hat{\mu}_{i,1} & 2 & \mu_{i,2} & & & \\ & \hat{\mu}_{i,3} & \ddots 2 & \mu_{i,3} & & \\ & & \ddots \hat{\mu}_{i,m-1} & \ddots 2 & \ddots \mu_{i,m-1} \\ & & & \hat{\mu}_{i,m} & 2 \end{bmatrix}_{(m\times m)}, \quad i = 1, 2, \cdots, m;$$

We also found that

$$(\overline{G}_3 + rI) = \text{diag}(H_1, H_2, \cdots, H_{m-1}, H_m)_{(m^2\times m^2)},$$

where

$$H_i = \text{diag}(r_1, \hat{H}_{i,2}, \hat{H}_{i,4}, \cdots, \hat{H}_{i,m-1})_{(m\times m)}, \quad j = 1, 3, \cdots, m(\text{odd}),$$

and

$$H_i = \text{diag}(\hat{H}_{i,1}, \hat{H}_{i,3}, \cdots, \hat{H}_{i,m-2}, r_1)_{(m\times m)}, \quad j = 2, 4, \cdots, m-1(\text{even}),$$

with

$$\hat{H}_{i,j} = \begin{bmatrix} r_1 & \mu_{i,j} \\ \hat{\mu}_{i,j+1} & r_1 \end{bmatrix}, \quad j = 1, 2, \cdots, m-1 \ .$$

Denoting $P_i \equiv H_i$ but with the diagonal element r_1 replaced by 1, equation (4.26) then becomes

$$\underline{\omega}_{(c)}^{(p+3/4)} = H_i^{-1}(P_i \underline{\omega}_{i(c)}^{(p)} + r\underline{\omega}_{i(c)}^{(p+1/2)}), \quad i = 1, 2, \cdots, m,$$

where

$$H_i^{-1} = \text{diag}(\frac{1}{r_1}, (\hat{H}_{i,2})^{-1}, (\hat{H}_{i,4})^{-1}, \cdots, (\hat{H}_{i,m-1})^{-1})_{(m\times m)}, \quad i = 1, 3, \cdots, m;$$

and

$$H_i^{-1} = \text{diag}(\ (\hat{H}_{i,1})^{-1}, (\hat{H}_{i,3})^{-1}, \cdots, (\hat{H}_{i,m-1})^{-1}, \frac{1}{r_1}\ )_{(m\times m)}, \quad i = 2, 4, \cdots, m-1;$$

with

$$(\hat{H}_{i,j})^{-1} = \frac{1}{\triangle_{i,j}} \begin{bmatrix} r_1 & -\mu_{i,j} \\ -\hat{\mu}_{i,j+1} & r_1 \end{bmatrix},$$

and

$$\hat{\triangle}_{j,i} = r_1^2 - \mu_{i,j}\hat{\mu}_{i,j+1}.$$

This results in the following equations for the computation at the current intermediate level

$$\omega_{i,1}^{(p+3/4)} = (\omega_{i,1}^{(p)} + r\omega_{i,1}^{(p+1/2)})/r_1, \quad i = 1, 3, \cdots, m, \tag{4.27}$$

$$\left\{ \begin{array}{ll} \omega_{i,j}^{(p+3/4)} &= (r_1 v_{i,j} - \mu_{i,j}\overline{v}_{i,j})/\hat{\triangle}_{j,i}, \text{and} \\ \omega_{i,j+1}^{(p+3/4)} &= (-\mu_{i,j+1}v_{i,j} + r_1\overline{v}_{i,j})/\hat{\triangle}_{j,i}, \end{array} \right\}, \quad i = 1, 3, \cdots, m; j = 2, 4, \cdots, m-1 \ , \tag{4.28}$$

and

$$\left\{ \begin{array}{ll} \omega_{i,j}^{(p+3/4)} &= (r_1 v_{i,j} - \mu_{i,j}\overline{v}_{i,j})/\hat{\triangle}_{j,i}, \text{and} \\ \omega_{i,j+1}^{(p+3/4)} &= (-\mu_{i,j+1}v_{i,j} + r_1\overline{v}_{i,j})/\hat{\triangle}_{j,i}, \end{array} \right\}, \quad i = 2, 4, \cdots, m-1; j = 1, 3, \cdots, m-2 \ , \tag{4.29}$$

$$\omega_{i,m}^{(p+3/4)} = (\omega_{i,m}^{(p)} + r\ \omega_{i,m}^{(p+1/2)})/r_1, \quad i = 2, 4, \cdots, m-1, \tag{4.30}$$

where

$$v_{i,j} = \omega_{i,j}^{(p)} + \mu_{i,j}\omega_{i,j+1}^{(p)} + r\omega_{i,j}^{(p+1/2)},$$
$$\overline{v}_{i,j} = \omega_{i,j+1}^{(p)} + \hat{\mu}_{i,j+1}\omega_{i,j}^{(p)} + r\omega_{i,j+1}^{(p+1/2)}.$$

(iv) At the $(p+1)^{\text{th}}$ iterate

Equation (4.9) is now transformed to

$$(\overline{G}_4 + rI)\underline{\omega}_{(c)}^{(p+1)} = \overline{G}_4\underline{\omega}_{(c)}^{(p)} + r\underline{\omega}_{(c)}^{(p+3/4)},$$

or

$$\underline{\omega}_{(c)}^{(p+1)} = (\overline{G}_4 + rI)^{-1}(\overline{G}_4\underline{\omega}_{(c)}^{(p)} + r\underline{\omega}_{(c)}^{(p+3/4)}), \tag{4.31}$$

we have

$$(\overline{G}_4 + rI) = \text{diag}(\overline{H_1}, \overline{H}_2, \cdots, \overline{H}_{m-1}, \overline{H}_m)_{(m^2\times m^2)}),$$

where

$$\overline{H}_i = \text{diag}(\hat{H}_{i,1}, \hat{H}_{i,3}, \cdots, \hat{H}_{i,m-2}, r_1)_(m \times m), \quad i = 1, 3, \cdots, m(\text{odd}),$$

and

$$\overline{H}_i = \text{diag}(r_1, \hat{H}_{i,2}, \hat{H}_{i,4}, \cdots, \hat{H}_{i,m-1})_(m \times m), \quad i = 2, 4, \cdots, m(\text{even}).$$

Denoting $Q_i \equiv \overline{H}_i$ (with the diagonal element r_1 replaced by 1), equation (4.31) can be written as

$$\underline{\omega}_{i(c)}^{(p+1)} = (\overline{H}_i)^{-1}(Q_i\underline{\omega}_{i(c)}^{(p)} + r\underline{\omega}_{i(c)}^{(p+3/4)}, \quad i = 1, 2, \cdots, m,$$

and as at the previous iterate, we obtain the following equations for computation

$$\omega_{i,j}^{(p+1)} = (r_1 z_{i,j} - \mu_{i,j}\overline{z}_{i,j})/\hat{\triangle}_{j,i}, \tag{4.32}$$

$$\omega_{i,j+1}^{(p+1)} = (-\hat{\mu}_{i,j+1}z_{i,j} + r_1\overline{z}_{i,j})/\triangle_{j,i}, \quad i = 1, 3, \cdots, m, j = 1, 3, \cdots, m-2 \tag{4.33}$$

$$\omega_{i,m}^{(p+1)} = (\omega_{i,m}^{(p)} + r\omega_{i,m}^{(p+3/4)})/r_1, \quad i = 1, 3, \cdots, m\ , \tag{4.34}$$

and

$$\omega_{i,1}^{(p+1)} = (\omega_{i,1}^{(p)} + r\omega_{i,1}^{(p+3/4)})/r_1, \quad i = 2, 4, \cdots, m-1, \tag{4.35}$$

$$\omega_{i,j}^{(p+1)} = (r_1 z_{i,j} - \mu_{i,j}\overline{z}_{i,j})/\hat{\triangle}_{j,i}, \tag{4.36}$$

$$\omega_{i,j+1}^{(p+1)} = (-\hat{\mu}_{i,j+1}z_{i,j} + r_1\overline{z}_{i,j})/\hat{\triangle}_{j,i}, \quad i = 2, 43, \cdots, m-1, j = 2, 4, \cdots, m-1\ , \tag{4.37}$$

where

$$z_{i,j} = \omega_{i,j}^{(p)} + \mu_{i,j}\omega_{i,j+1}^{(p)} + r\omega_{i,j}^{(p+3/4)},$$
$$\overline{z}_{i,j} = \omega_{i,j+1}^{(p)} + \hat{\mu}_{i,j+1}\omega_{i,j}^{(p)} + r\omega_{i,j+1}^{(p+3/4)}.$$

Hence, we write $\omega_{ij}^{(p+1/4)}, \omega_{ij}^{(p+1/2)}, \omega_{ij}^{(p+3/4)}, \omega_{ij}^{(p+1)}$ in an explicit equations. These equations are listed in the following table.

TABLE 4.1
Equations for all four intermediate steps

Intermediate step	Equation Number
$\omega_{ij}^{p+1/4}$	(4.14), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20)
$\omega_{ij}^{p+1/2}$	(4.22), (4.23), (4.24), (4.25)
$\omega_{ij}^{p+3/4}$	(4.27), (4.28), (4.29), (4.30)
ω_{ij}^{p+1}	(4.32), (4.33), (4.34), (4.35), (4.36), (4.37)

5. NUMERICAL ALGORITHM. From Section(4), an algorithm can now be formulated to solve the equation (2.2). Given all the data of the problem and an initial approximation ω_{ij}^{old}, INNER-AGE ALGORITHM will compute a better approximation ω_{ij}^{new}.

After we state the INNER-AGE ALGORITHM, we are ready to solve equations (2.1) and (2.2). This is described in OUTER-AGE ALGORITHM which will compute an approximation for the streamfunction ψ and the vorticity ω.

Algorithm 1 (INNER-AGE ALGORITHM)

Given: β_{ij}, $\hat{\beta}_{ij}$, μ_{ij}, $\hat{\mu}_{ij}$, f_{ij}, Re, ϵ_1, q, m, r,
ϵ *and* ω_{ij}^{old}, This algorithm computes ω_{ij}^{new}

p=0
$\omega_{ij}^{(p)} = \omega_{ij}^{old}$
repeat
step $\frac{1}{4}$**:** Compute $\omega_{ij}^{(p+1/4)}$
by using equation (4.14), (4.15), (4.16), (4.17), (4.18), (4.19),(4.20)
step $\frac{1}{2}$**:** Compute $\omega_{ij}^{(p+1/2)}$
by using equation (4.22), (4.23), (4.24), (4.25)
step $\frac{3}{4}$**:** Compute $\omega_{ij}^{(p+3/4)}$
by using equation (4.27), (4.28), (4.29), (4.30)
step 1**:** Compute $\omega_{ij}^{(p+1)}$
by using equation (4.32), (4.33), (4.34), (4.35), (4.36), (4.37)
step 2**:** Compute $\tau = \max_{i,j} \{ \mid \omega_{ij}^{(p+1)} - \omega_{ij}^{(p)} \mid \}$ and set $p = p + 1$
until $(\tau < \epsilon)$
$\omega_{ij}^{(new)} = \omega_{ij}^{(p)}$

Algorithm 2 (OUTER-AGE ALGORITHM)

Given:
- problem parameters: Re, ϵ_1, q
- acceleration-parameter: r
- mesh-size: m
- inner-convergence-criterion: ϵ
- outer-convergence criterion: δ

This algorithm computes an approximation for ψ and ω

Set $k = 0$ and $h = \frac{1}{m+1}$
Set $\psi_{ij}^{(k)} = 0$ and $\omega_{ij}^{(k)} = 0$ as initial approximations
Compute f_{ij} using equation (4.1)
repeat
Compute $\psi_{ij}^{(k+1)}$ using INNER-AGE ALGORITHM
[Here, $Re = 0$, $\psi_{ij}^{(k)}$ replaces ω_{ij}^{old}, $\psi_{ij}^{(k+1)}$ replaces ω_{ij}^{new}, $f_{ij} = h^2\omega_{ij}^{(k)}$]
Compute $\tau = \max_{i,j} \{ \mid \psi_{ij}^{(k+1)} - \psi_{ij}^{(k)} \mid, \mid \omega_{ij}^{(k-1)} - \omega_{ij}^{(k)} \mid \}$ (do this step if $k > 0$)
Compute $\tilde{A}_{ij}^{(k+1)}$ using equation (3.4)
Compute $\beta_{ij}, \hat{\beta}_{ij}, \mu_{ij}, \hat{\mu}_{ij}$ using equation (4.2, 4.3, 4.4, 4.5)
Compute $\omega_{ij}^{(k+1)}$ using INNER-AGE ALGORITHM
Compute $\tau = \max_{i,j} \{ \mid \psi_{ij}^{(k+1)} - \psi_{ij}^{(k)} \mid, \mid \omega_{ij}^{(k+1)} - \omega_{ij}^{(k)} \mid \}$ and set $k = k + 1$
until $(\tau < \delta)$
$\omega_{ij}^{(new)} = \omega_{ij}^{(p)}$

6. NUMERICAL EXAMPLES. This section presents the results of numerical experiment with the (AGE) iterative algorithm described in section OUTER-AGE ALGORITHM. Specifically, we consider one example with known exact solution. This example has been studied in [8, 16]. For this section, the region Ω is the unit square $0 < x < 1,\ \ 0 < y < 1$ and the exact solution is

$$\psi^*(x,y) = x^2(x-1)^2y^2(y-1)^2,$$
$$\omega^*(x,y) = -\triangle\,\psi^*(x,y).$$

Example 1:
We consider the following equations

$$\triangle^2\psi = -w \quad \text{in} \quad \Omega,$$
$$\triangle(A(\psi)w) + Re(\psi_x w_y - \psi_y w_x) = -g \quad \text{on} \quad \partial\Omega,$$

subject to the boundary conditions

$$(x,0) = \psi(x,1) = w(x,0) = w(x,1) = 0, \quad 0 \le x \le 1,$$
$$\psi(0,y) = \psi(1,y) = w(0,y) = w(1,y) = 0, \quad 0 \le y \le 1,$$

where the function g is defined as

$$g = \triangle(\tilde{A}(\psi^*)\,\triangle\,\psi^*) - Re(\psi^*_y\,\triangle\,\psi^*_x - \psi^*_x\,\triangle\,\psi^*_y).$$

The value of the Reynolds number is $Re = 50$ with the following parameters $\epsilon_1 = 10^{-20}, m = 29, h = \frac{1}{30}, q = 4$, and $r = 0.8$. The termination criteria for the outer and inner iteration, i.e., δ and ϵ are chosen as $\delta = 10^{-5}, \epsilon = 10^{-13}$. the number of iterations required to attain convergence is 2. The numbers of inner iterations required for the first outer iteration are 1, 489. The numbers of inner iterations required for the second outer iteration are 436, 0. Table(6.1) and Table(6.3) display the values of the exact solution ψ^* and the computed values of the streamfunction ψ^h. Table(6.2) and Table(6.4) display the values of the exact solution ω^* and the computed values of the vorticity ω^h. A quick comparison between Table(6.1) Table(6.3) shows an agreement. Similarly, Table(6.2) and Table(6.4) also shows an agreement. The values in Table(6.3) and Table(6.4) are good approximations to the exact solution. Figure(6.1(a)) and Figure(6.1(c)) show the contours for the exact solution ψ^* and the computed solution ψ^h. Figure(6.1(b)) and figure(6.1(d)) show the contours for the exact solution ω^* and the computed solution ω^h. From Figure(6.1), we can see a good agreement between each corresponding graphs.

Table 6.1
Exact values of ψ^* at (x,y) where $x, y = 0.1, 0.3, 0.5, 0.7, 0.9$

ψ^*	.962361E-02	.249272E-01	.306563E-01	.249272E-01	.962361E-02
	.249272E-01	.645668E-01	.794062E-01	.645668E-01	.249272E-01
	.306563E-01	.794062E-01	.976563E-01	.794063E-01	.306563E-01
	.249272E-01	.645668E-01	.794063E-01	.645668E-01	.249272E-01
	.962361E-02	.249272E-01	.306563E-01	.249272E-01	.962361E-02

Table 6.2
Exact values of ω^* at (x,y) where $x, y = 0.1, 0.3, 0.5, 0.7, 0.9$

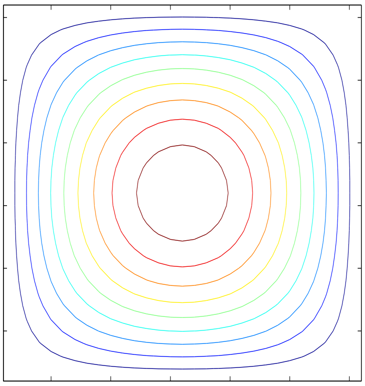
ω^*	.211896E+00	.521640E+00	.631800E+00	.521640E+00	.211896E+00
	.521640E+00	.128066E+01	.154980E+01	.128066E+01	.521640E+00
	.631800E+00	.154980E+01	.187500E+01	.154980E+01	.631800E+00
	.521640E+00	.128066E+01	.154980E+01	.128066E+01	.521640E+00
	.211896E+00	.521640E+00	.631800E+00	.521640E+00	.211896E+00

TABLE 6.3
Computed values of ψ^h *at* (x, y) *where* $x, y = 0.1, 0.3, 0.5, 0.7, 0.9$

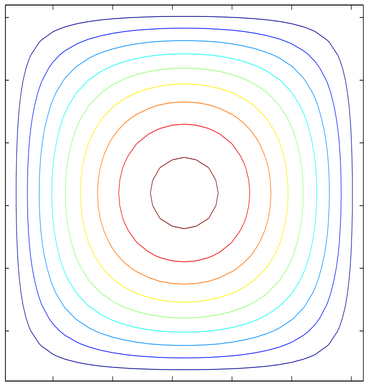
	.963875E-02	.248545E-01	.307007E-01	.250736E-01	.963875E-02
	.250736E-01	.646559E-01	.795136E-01	.646559E-01	.248545E-01
ψ^h	.307007E-01	.795136E-01	.977855E-01	.795136E-01	.307007E-01
	.248545E-01	.646559E-01	.795136E-01	.646559E-01	.250736E-01
	.963875E-02	.250736E-01	.307007E-01	.248545E-01	.963875E-02

TABLE 6.4
Exact values of ω^h *at* (x, y) *where* $x, y = 0.1, 0.3, 0.5, 0.7, 0.9$

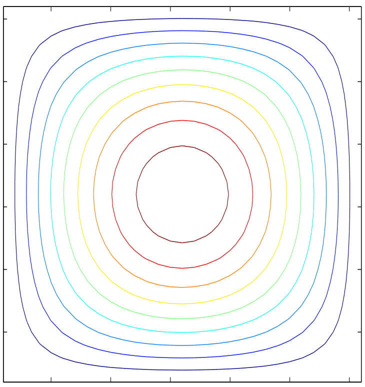
	.212004E+00	.498832E+00	.632100E+00	.544952E+00	.212004E+00
	.544952E+00	.128125E+01	.155050E+01	.128125E+01	.498832E+00
ω^h	.632100E+00	.155050E+01	.187583E+01	.155050E+01	.632100E+00
	.498832E+00	.128125E+01	.155050E+01	.128125E+01	.544952E+00
	.212004E+00	.544952E+00	.632100E+00	.498832E+00	.212004E+00



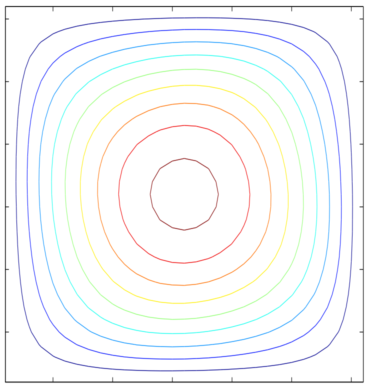
(a) Contours of the exact solution ψ^*



(b) Contours of the exact solution ω^*



(c) Contours of the computed solution ψ^h



(d) Contours of the computed solution ω^h

FIG. 6.1. *Contours of the exact and computed solutions*

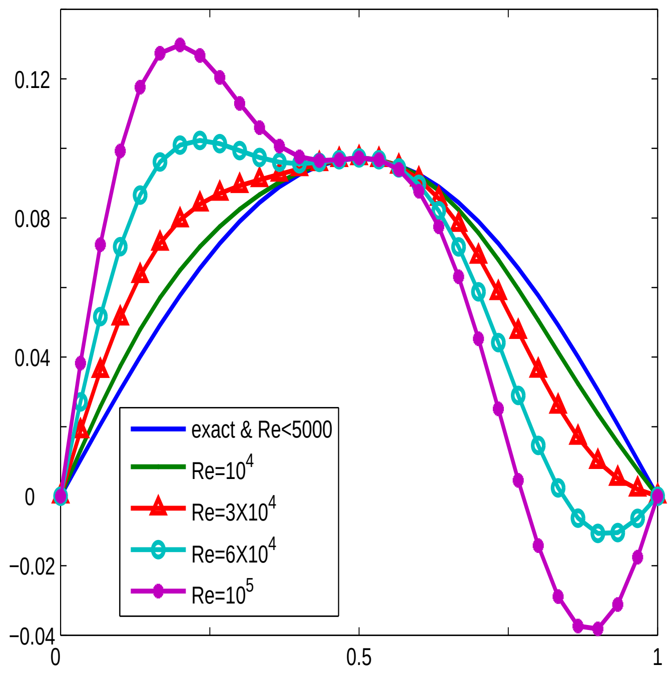


FIG. 6.2. *ψ^h-streamfunction lines at different values of the Reynolds numbers through the horizontal line $y = 0.5$.*

Example 2:
We consider the same problem in Example(1) with the following parameters:

$$\epsilon_1 = 10^{-3}, q = 4, m = 29, r = 0.8 \ ,$$
$$\epsilon = 10^{-5}, \delta = 10^{-13}, h = \frac{1}{30}.$$

We compute an approximate solution for $Re = 1,\ 10,\ 10^2,\ 10^3,\ 5\times10^3,\ 10^4,\ 3\times10^4,\ 6\times10^4,\ 10^5$. Figure(6.2) displays the plot of the streamfunction along the vertical line $x = \frac{1}{2}$ passing through the point $(0.5, 0.5)$ with the above values for Reynolds numbers. Also, the numerical programs were performed for a series of different values of the Reynolds numbers between 1 and 1000. Each time, we evaluate the difference between the exact solution ψ^* and the computed solution. Then, we plot the graph in Figure(6.3). The horizontal axis of the graph represents $\log_{10}(Re)$ and the vertical axis represents $\|\ \psi^* - \psi^{computed}\ \|_{L_2}$. We can see more clearly the fact that the difference in the discrete norm increases as Re increases.

Example 3:
We conducted convergence tests to obtained error estimates and assess the order of accuracy. We solve the same problem in Example (1) with different mesh size h. The parameters of the problem are:

$$\epsilon_1 = 10^{-20}, q = 4, r = 0.8, Re = 1.0 \ .$$

In Table (6.5) we show the discrete L_2 norm $\|\ \psi^* - \psi^h\ \|_{L_2}$ where ψ^h denotes the computational solution on an $m\times m$ grid ($h = \frac{1}{m+1}$). The convergence rates are computed using the information on two successive meshes.

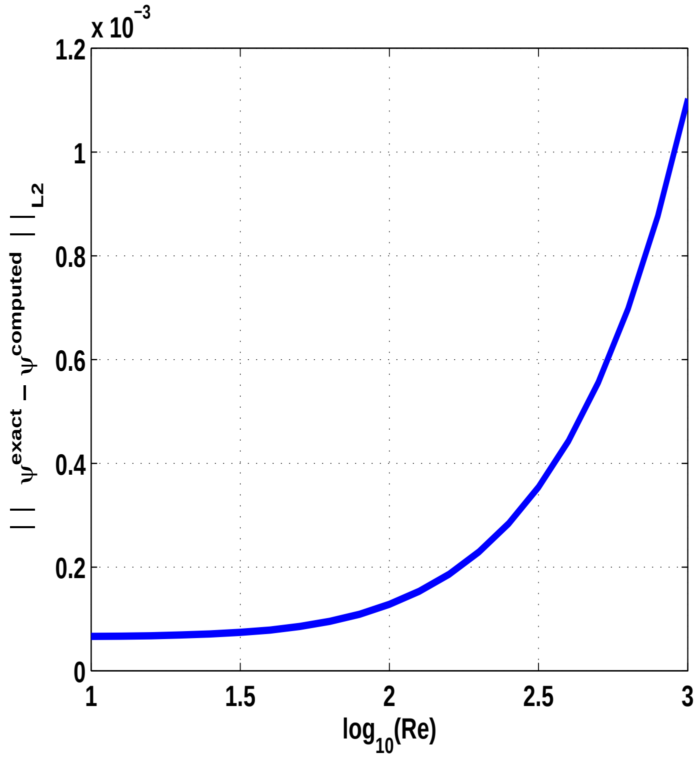


FIG. 6.3. *Difference between the exact solution* ψ^* *and the computed solution* ψ^h *.vs.* $\log_{10}(Re)$.

TABLE 6.5
Error estimate and order of convergence

m	h	$\|\| \psi^* - \psi^h \|\|_{L_2}$	order of convergence
9	$\frac{1}{10}$	1.9647844e-004	
13	$\frac{1}{14}$	1.4015979e-004	1.0038
19	$\frac{1}{20}$	9.8045095e-005	1.0019
23	$\frac{1}{24}$	8.1689733e-005	1.0009
29	$\frac{1}{30}$	6.5346024e-005	1.0003

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