Faber and Manteuffel, 1984

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A CONJUGATE GRADIENT METHOD*

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Abstract. We characterize the class $CG(s)$ of matrices A for which the linear system $A\mathbf{x}=\mathbf{b}$ can be solved by an s-term conjugate gradient method. We show that, except for a few anomalies, the class $CG(s)$ consists of matrices A for which conjugate gradient methods are already known. These matrices are the Hermitian matrices, $A^*=A$, and the matrices of the form $A=e^{i\theta}(dI+B)$, with $B^*=-B$.

- Faber and Manteuffel gave the answer in 1984:
  For a general matrix $\mathbf{A}$ there exists *no* short recurrence for generating orthogonal Krylov subspace bases.
- **What are the details of this statement**?

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Basic question

What are sufficient and necessary conditions for $\mathbf{A}$ to admit an optimal $(s+2)$-term recurrence?

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Definition. If

$$\mathbf{A}^* = p_s(\mathbf{A}),$$

where p_s is a polynomial of the smallest possible degree s, $\mathbf{A}$ is called normal(s).

Outline

The Faber-Manteuffel theorem

Theorem. [Faber and Manteuffel, 1984], [Liesen and Strakoš, 2008]

Let $\mathbf{A}$ be a nonsingular matrix with minimal polynomial degree $d_{\min}(\mathbf{A})$. Let s be a nonnegative integer, $s + 2 < d_{\min}(\mathbf{A})$:

$\mathbf{A}$ admits an optimal $(s + 2)$-term recurrence

if and only if

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- Sufficiency is rather straightforward, necessity *is not*. Key words from the proof of necessity in (Faber and Manteuffel, 1984) include: "continuous function" (analysis), "closed set of smaller dimension" (topology), "wedge product" (multilinear algebra).

Summary

Generating of orthogonal basis of $\mathcal{K}_d(\mathbf{A}, v)$ via short recurrences

Arnoldi-type recurrence $(s+2)$-term

$$\Updownarrow$$

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  1. $s = 1$ if and only if the eigenvalues of $\mathbf{A}$ lie on a line in $\mathbb{C}$.
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- All classes of "interesting" matrices are known.

Related papers

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  Completely reworked theory of short recurrences for generating orthogonal Krylov subspace bases

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  New proofs of the fundamental theorem of Faber and Manteuffel

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Thank you for your attention!