On a New Proof of the Faber-Manteuffel Theorem

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joint work with

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Outline

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Krylov subspace methods

Given $\mathbf{A} \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n$. Define the jth Krylov subspace

$$\mathcal{K}_j(\mathbf{A}, v) \equiv \operatorname{span}(v, \mathbf{A}v, \ldots, \mathbf{A}^{j-1}v)\,.$$

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Krylov subspace methods:

- Iterative methods for solving large and sparse linear systems or eigenvalue problems,
- they are based on projection onto the Krylov subspaces,
- examples: Lanczos, CG, Arnoldi, GMRES, BiCG.

Krylov subspace methods

Basis

Each method must generate a basis of $\mathcal{K}_j(\mathbf{A}, v), \quad j = 1, 2, \ldots$

- The trivial choice $v, \mathbf{A}v, \ldots, \mathbf{A}^{j-1}v$ is computationally infeasible (recall the Power Method).

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- For computational efficiency: Short recurrence.

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- For numerical stability: Well conditioned basis.
- For computational efficiency: Short recurrence.
- Best of both worlds:
  Orthogonal basis computed by short recurrence.

Optimal Krylov subspace methods

with short recurrences

CG (1952), MINRES, SYMMLQ (1975)

- based on three-term recurrences

$$r_{j+1} = \gamma_j \mathbf{A} r_j - \alpha_j r_j - \beta_j r_{j-1} \,,$$

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- generate orthogonal (or $\mathbf{A}$-orthogonal) Krylov subspace basis,

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- *optimal* in the sense that they minimize some error norm:

  $\|x - x_j\|_{\mathbf{A}}$ in CG,

  $\|x - x_j\|_{\mathbf{A}^T\mathbf{A}} = \|r_j\|$ in MINRES,

  $\|x - x_j\|$ in SYMMLQ -here $x_j \in x_0 + \mathbf{A}\mathcal{K}_j(\mathbf{A}, r_0)$.

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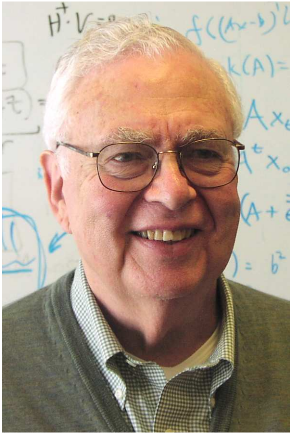
- An important assumption on $\mathbf{A}$:
  $\mathbf{A}$ is symmetric (MINRES, SYMMLQ) & pos. definite (CG).

Gene Golub



G. H. Golub, 1932–2007

- By the end of the 1970s it was unknown if such methods existed also for general unsymmetric $\mathbf{A}$.
- Gatlinburg VIII (now Householder VIII) held in Oxford from July 5 to 11, 1981.
- "A prize of $500 has been offered by Gene Golub for the construction of a 3-term conjugate gradient like descent method for non-symmetric real matrices or a proof that there can be no such method".