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H^2 solutions for the stream function and vorticity formulation of the Navier–Stokes equations

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Abstract

We show that the two dimensional Navier–Stokes equations in the stream function and vorticity form with nonhomogeneous boundary conditions have a unique solution with a stream function having two space derivatives.

Keywords: Stream function; Vorticity; Nonhomogeneous Navier–Stokes equations

1. Introduction

Most of the mathematical literature addressing the analysis of the Navier–Stokes equations has dealt with the case of homogeneous boundary conditions. The velocity and pressure formulation received much more attention than the stream function and vorticity formulation. The available mathematical results can mostly be found in [3,4,7] and in the more recent work in [2].

A variety of mathematical formulations have been studied to understand the Navier–Stokes equations from analytical as well as numerical point of view. In this work the stream function and vorticity formulation are studied. This formulation is commonly used in the engineering literature (see for example [1]). Some related studies of the stationary problem are found in [5,6].

The objective of this work is to examine the existences and uniqueness problems of the Navier–Stokes equations with nonhomogeneous boundary conditions using the stream function and vorticity formulation. Here we take the stream function to be in a space of more smoothness conditions than the traditionally taken of only one weak derivative.

The rest of this paper consists of four sections. Section 2 presents the classical formulation. The properties of a bilinear operator B that appears in the formulation are discussed in Section 3. An equivalent variational

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formulation is given in Section 4. Finally, our results on the existence and uniqueness of solution of the variational problem are given in Section 5.

2. Problem formulation

In terms of the stream function ψ and vorticity φ, the equations governing a time dependent viscous flow are given by

$$\begin{aligned} &\frac{\partial\varphi}{\partial t} - \nu\Delta\varphi - \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x} = q, \\ &-\Delta\psi = \varphi \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where ν denotes the kinematic viscosity, $q = \operatorname{curl}\mathbf{f}$, $\mathbf{f}$ is the body force and Ω is a bounded region in $\mathbb{R}^2$ with finite, Lipschitz continuous boundary $\partial\Omega$. We assume that $\partial\Omega$ consists of two parts Γ_1, Γ_2 with positive $\mathbb{R}^1$ measure and empty intersection of the interiors.

All treatments of (1) assume at least one derivative of the vorticity function φ. The second equation means that the stream function ψ should have two derivatives. We intend to study the solutions of (1) under the assumption that ψ has at least two derivatives. For this purpose, we rewrite the second equation of (1) as

$$-\Delta^2\psi = \Delta\varphi, \tag{2}$$

where the derivatives involved here are understood to be in a weak sense as clarified later.

The following boundary conditions will be considered:

$$\begin{aligned} &\psi = \frac{\partial\psi}{\partial\mathbf{n}} = 0, \quad \text{on } \Gamma_1, \\ &\psi = g_1, \ \frac{\partial\psi}{\partial\mathbf{n}} = g_2, \ \varphi = 0 \quad \text{on } \Gamma_2, \end{aligned} \tag{3}$$

where the functions g_1, g_2 are assumed to be compatible with the existence of a stream function (see [4]) and $g_1 \in H^{3/2}(\Gamma_2)$, $g_2 \in H^{1/2}(\Gamma_2)$. Initial conditions on the vorticity function φ are specified as

$$\varphi(0) = \varphi_0 \quad \forall (x,y) \in \Omega. \tag{4}$$

In the sequel, the space $L^2(\Omega)$ will be denoted by H and its norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$ respectively and we introduce a formal bilinear operator B by

$$B(\varphi,\psi) = -\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}.$$

With this operator, Eqs. (1), and (2) take the form

$$\begin{aligned} &\frac{\partial\varphi}{\partial t} - \nu\Delta\varphi + B(\varphi,\psi) = q, \\ &-\Delta^2\psi = \Delta\varphi. \end{aligned} \tag{5}$$

Our starting point will be to transform the problem (3)–(5) to an equivalent one with homogeneous boundary conditions. For this purpose we let ψ_b be the unique solution in $H^2(\Omega)$ of

$$\Delta^2\psi_b = 0, \qquad \begin{aligned} &\psi_b = g_1, \ \frac{\partial\psi_b}{\partial\mathbf{n}} = g_2 \quad \text{on } \Gamma_2, \\ &\psi_b = \frac{\partial\psi_b}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{6}$$

With the transformation $\psi \mapsto \psi - \psi_b$ the system (3)–(5) changes into

$$\begin{aligned} &\frac{\partial\varphi}{\partial t} - \nu\Delta\varphi + B(\varphi,\psi) - B(\varphi,\psi_b) = q, \\ &\Delta^2\psi + \Delta\varphi = 0 \quad \text{in } \Omega, \\ &\varphi(0) = \varphi_0, \end{aligned}$$

$$\psi = \frac{\partial \psi}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$
$$\varphi = 0 \quad \text{on } \Gamma_2. \tag{7}$$

Note that no boundary values are assumed for the vorticity function φ on Γ_1.

The main function spaces we need here are

$$V_\psi = H_0^2(\Omega),$$
$$V_\varphi = \{\zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \Gamma_2\}.$$

In addition to these we also use the standard Bochner spaces such as $L^2(0,T;X)$, $\mathscr{H}^\gamma(0,T,X,Y)$, ..., etc., the definitions and properties of which can be found in [4].

3. Properties of the bilinear operator B

In this section and the rest of the paper, c will denote a generic constant that may depend on the properties of the domain Ω but is independent of the functions involved, and may vary from inequality to another, $\mathscr{D}(\Omega)$ will denote the space of test functions consisting of infinitely differentiable compactly supported (in Ω) functions, and V' will denote the dual space, of a space V, consisting of generalized functions.

We now discuss some properties of the bilinear operator B that will be used extensively in this work. To begin with we notice that the operator B has the simple property

$$B(\eta, \eta) = 0.$$

Further properties of B are given in the following lemmas.

Lemma 1. *V_φ is continuously embedded in $L^4(\Omega)$. Moreover, we have the inequality*

$$\|\chi\|_4 \leqslant c\|\nabla\chi\|^{1/2}\|\chi\|^{1/2},$$

for every $\chi \in V_\varphi$, where $\|\cdot\|_4$ denotes the norm in $L^4(\Omega)$.

Proof. See [4, p. 156]. □

Lemma 2. *V_ψ is continuously embedded in $L^4(\Omega)$. Moreover, we have the inequality*

$$\|\nabla\chi\|_4 \leqslant c\|\Delta\chi\|^{1/2}\|\nabla\chi\|^{1/2}$$

for every $\chi \in V_\psi$. Here, $\|\nabla\chi\|_4 = \left(\int_\Omega \left|\frac{\partial\chi}{\partial x}\right|^4 + \left|\frac{\partial\chi}{\partial y}\right|^4 \mathrm{d}\Omega\right)^{1/4}$.

Proof. Assume first that $\chi \in \mathscr{D}(\Omega)$. Since $\frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y} \in V_\varphi$, we have by Lemma 1

$$\int \left|\frac{\partial\chi}{\partial x}\right|^4 \leqslant c\left\|\nabla\frac{\partial\chi}{\partial x}\right\|^2\left\|\frac{\partial\chi}{\partial x}\right\|^2,$$
$$\int \left|\frac{\partial\chi}{\partial y}\right|^4 \leqslant c\left\|\nabla\frac{\partial\chi}{\partial y}\right\|^2\left\|\frac{\partial\chi}{\partial y}\right\|^2.$$

On the other hand,

$$\left\|\nabla\frac{\partial\chi}{\partial x}\right\|^2 = \int\left|\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial x\,\partial y}\right|^2 \leqslant c\left(\int\left(\frac{\partial^2\chi}{\partial x^2}\right)^2 + \int\left(\frac{\partial^2\chi}{\partial x\,\partial y}\right)^2\right) \leqslant c|\chi|_{2,\Omega}^2 = c\|\Delta\chi\|^2.$$

Similarly,

$$\left\|\nabla\frac{\partial\chi}{\partial y}\right\|^2 \leqslant c\|\Delta\chi\|^2.$$

Hence,

$$\|\nabla\chi\|_4^4 \leqslant c\|\Delta\chi\|^2\left(\left\|\frac{\partial\chi}{\partial x}\right\|^2+\left\|\frac{\partial\chi}{\partial y}\right\|^2\right)=c\|\Delta\chi\|^2\|\nabla\chi\|^2.$$

The result now follows by the density of $\mathscr{D}(\Omega)$ in V_ψ. □

Lemma 3. $\langle B(\varphi,\psi),\chi\rangle = -\langle B(\chi,\psi),\varphi\rangle$, $\forall\varphi,\chi\in H^1(\Omega)$, $\psi\in V_\psi$.

Proof. It suffices to prove the statement for $\varphi,\chi\in C^\infty(\Omega)$, $\psi\in\mathscr{D}(\Omega)$.

$$\begin{aligned}\langle B(\varphi,\psi),\chi\rangle &= \int\left(-\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)\chi=\int\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\chi-\int\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}\chi\\ &=-\int\varphi\left(\frac{\partial^2\psi}{\partial y\partial x}\chi+\frac{\partial\psi}{\partial x}\frac{\partial\chi}{\partial y}\right)+\int\varphi\left(\frac{\partial^2\psi}{\partial x\partial y}\chi+\frac{\partial\chi}{\partial y}\frac{\partial\psi}{\partial x}\right)\\ &=-\int\left(-\frac{\partial\chi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\chi}{\partial y}\frac{\partial\psi}{\partial x}\right)\varphi\\ &=-\langle B(\chi,\psi),\varphi\rangle. \qquad\square\end{aligned}$$

Lemma 4. $B: V_\varphi\times V_\psi\to V'_\varphi$ *continuously. Moreover,*

$$\|B(\varphi,\psi)\|_{V'_\varphi}\leqslant c\|\nabla\varphi\|\|\Delta\psi\|. \tag{8}$$

Proof. For φ, $\chi\in V_\varphi$, $\psi\in\mathscr{D}(\Omega)$ we have, using Lemmas 1 and 2 and the inequality $\|\nabla\psi\|\leqslant c\|\Delta\psi\|$,

$$\begin{aligned}|\langle B(\varphi,\psi),\chi\rangle| &= \left|\int\left(-\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)\chi\right|\leqslant\int|\nabla\varphi||\nabla\psi||\chi|\leqslant\|\nabla\varphi\|\|\nabla\psi\|_4\|\chi\|_4\\ &\leqslant c\|\nabla\varphi\|\|\nabla\psi\|_4\|\nabla\chi\|^{1/2}\|\chi\|^{1/2}\leqslant c\|\nabla\varphi\|\|\nabla\psi\|_4\|\nabla\chi\|\leqslant c\|\nabla\varphi\|\|\Delta\psi\|^{1/2}\|\nabla\psi\|^{1/2}\|\nabla\chi\|\\ &\leqslant c\|\nabla\varphi\|\|\Delta\psi\|\|\nabla\chi\|. \qquad\square\end{aligned}$$

Corollary 5. *Suppose that* $\psi\in V_\psi$ *and* $\varphi\in V_\varphi$ *satisfy* $\varphi=\Delta\psi$, *then*

$$|\langle B(\varphi,\psi),\chi\rangle|\leqslant c\|\nabla\chi\|\|\nabla\varphi\|^2.$$

Proof. This follows from Lemma 4 and the inequality $\|\varphi\|\leqslant c\|\nabla\varphi\|$. □

Lemma 6. $B(\cdot,\psi_b): V_\varphi\to V'_\varphi$ *continuously. Furthermore,* $\|B(\varphi,\psi_b)\|_{V'_\varphi}\leqslant\|\nabla\varphi\|\|\psi_b\|_\infty$.

Proof. By restricting $\varphi\in V_\varphi$ to functions with continuous two derivatives that vanish in a neighborhood of Γ_2, taking into account the boundary conditions of ψ_b and the continuous embedding $H^2(\Omega)\hookrightarrow C(\overline{\Omega})$, we have, for all $\chi\in V_\varphi$,

$$|\langle B(\varphi,\psi_b),\chi\rangle|=|\langle B(\varphi,\chi),\psi_b\rangle|\leqslant\|\nabla\varphi\|\|\nabla\chi\|\|\psi_b\|_\infty. \tag{9}$$

The result follows by density. □

4. Variational formulation

In this section we give an equivalent variational formulation of (7). We will use the notations $\Phi(0,T)=L^2(0,T;V_\varphi)$, $\Psi(0,T)=L^2(0,T;V_\psi)$. The variational formulation of (7) is stated as follows:

For $q \in L^2(0,T;V'_\varphi)$, $\varphi_0 \in H$, find a pair (φ,ψ) with

$$\varphi \in \Phi(0,T) \cap L^\infty(0,T;H), \qquad \psi \in \Psi(0,T) \cap L^\infty\left(0,T;H_0^1(\Omega)\right)$$

such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi(t),\chi\rangle + v(\nabla\varphi(t),\nabla\chi) + \langle B(\varphi(t),\psi(t)),\chi\rangle + \langle B(\varphi(t),\psi_b),\chi\rangle = \langle q(t),\chi\rangle,\\ &\quad \forall \chi \in V_\varphi, \varphi(0) = \varphi_0,\end{aligned} \tag{10}$$

and

$$(\Delta\psi(t),\Delta\xi) - (\nabla\varphi(t),\nabla\xi) = 0 \quad \forall \xi \in V_\psi. \tag{11}$$

The system of Eqs. (10) and (11) will be referred to as Problem (P).

Theorem 7. *If* $\varphi \in \Phi(0,T) \cap L^\infty(0,T;H)$, $\psi \in \Psi(0,T) \cap L^\infty(0,T;H_0^1(\Omega))$ *solve Problem* (P) *then*

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^2\left(0,T;V'_\varphi\right).$$

Proof. For $\chi \in V_\varphi$ we have

$$\begin{aligned}|\langle B(\varphi(t),\psi(t)),\chi\rangle| &= |\langle B(\chi,\psi(t)),\varphi(t)\rangle| \leqslant c\|\nabla\chi\|\|\nabla\psi(t)\|_4\|\varphi(t)\|_4\\ &\leqslant c\|\nabla\chi\|\|\Delta\psi(t)\|^{1/2}\|\nabla\varphi(t)\|^{1/2}\|\nabla\psi(t)\|^{1/2}\|\varphi(t)\|^{1/2}\end{aligned}$$

so that

$$\left(\int_0^T \|B(\varphi(t),\psi(t))\|_{V_\varphi}^2\,\mathrm{d}t\right)^{1/2} \leqslant c\|\varphi\|_\Phi\|\psi\|_\Psi\|\psi\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)}\|\varphi\|_{L^\infty(0,T;H)},$$

i.e., $B(\varphi(\cdot),\psi(\cdot)) \in L^2(0,T;V'_\varphi)$.

Similarly, noting that ψ_b possesses enough smoothness to allow the switch in the first equation below, we have, for $\chi \in V_\varphi$ sufficiently smooth,

$$\begin{aligned}|\langle B(\varphi(t),\psi_b),\chi\rangle| &= |\langle B(\chi,\psi_b),\varphi(t)\rangle| \leqslant c\|\nabla\chi\|\|\nabla\psi_b\|_4\|\varphi(t)\|_4\\ &\leqslant c\|\nabla\chi\|\|\Delta\psi_b\|^{1/2}\|\nabla\varphi(t)\|^{1/2}\|\nabla\psi_b\|^{1/2}\|\varphi(t)\|^{1/2}\\ &\leqslant c\|\nabla\chi\|\|\psi_b\|_{H^2(\Omega)}\|\nabla\varphi(t)\|^{1/2}\|\varphi(t)\|^{1/2}\end{aligned}$$

so that

$$\left(\int_0^T \|B(\varphi(t),\psi_b)\|_{V_\varphi}^2\,\mathrm{d}t\right)^{1/2} \leqslant c\|\varphi\|_\Phi\|\psi_b\|_{H^2(\Omega)}\|\varphi_{L^\infty(0,T;H)},$$

which shows that $B(\varphi(\cdot),\psi_b) \in L^2(0,T;V'_\varphi)$. Our assumption that $q \in L^2(0,T;V'_\varphi)$ then gives $\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^2(0,T;V'_\varphi)$. □

Corollary 8. *Problem* (P) *is equivalent to* (7).

5. Existence, uniqueness for Problem (P)

In this section we show that Problem (P) has a unique solution. This is the content of Theorems 13 and 14 at the end of this section. Our main tool is a standard Galerkin argument. What may come as a surprise in this section is that the same argument used for the proof of existence and uniqueness of the Navier–Stokes equation in the velocity–pressure formulation can be modified to work for the stream function and vorticity one. This modification, however, is not trivial since it relies heavily on the properties of the underlying spaces and the operator B introduced in the previous two sections as well as on the isomorphic features of the operators Δ and Δ^2.

We begin by discretizing the space V_φ. Let $(\varphi_m)_{m\geqslant 1}$ be a basis for V_φ (i.e., V_φ is the closure of finite linear combinations of the functions φ_m). Let $(\psi_m)_{m\geqslant 1}$ be the unique solutions in V_ψ of

$$-\Delta^2\psi_m = \Delta\varphi_m \quad \text{in } V'_\psi.$$

Notice that, since in the mappings

$$V_\psi \xrightarrow{\Delta} H \xrightarrow{\Delta} H^{-2}(\Omega),$$

Δ is an isomorphism, we have

$$\Delta\psi_m = \varphi_m \quad \text{in } H. \tag{12}$$

Set $V_\varphi^m = \text{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$, $V_\psi^m = \text{span}(\psi_1, \psi_2, \ldots, \psi_m)$. Consider the following discretized problem: Find functions $(g_{1m}(t), g_{2m}(t), \ldots, g_{mm}(t))$ such that

$$\varphi_m(t) = \sum_{j=1}^{m} g_{jm}(t)\varphi_j, \tag{13}$$

$$\psi_m(t) = \sum_{j=1}^{m} g_{jm}(t)\psi_j, \tag{14}$$

are solutions of

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle\varphi_m(t), \varphi_i\rangle + \nu(\nabla\varphi_m(t), \nabla\varphi_i) + \langle B(\varphi_m(t), \psi_m(t)), \varphi_i\rangle + \langle B(\varphi_m(t), \psi_b), \varphi_i\rangle = \langle q, \varphi_i\rangle,\\ &\varphi_m(0) = \varphi_{0m} \in V_\varphi\end{aligned} \tag{15}$$

and

$$(\Delta\psi_m(t), \Delta\psi_i) + (\nabla\varphi_m(t), \nabla\psi_i) = 0, \quad i = 1, 2, \ldots, m, \tag{16}$$

where φ_{0m} is chosen such that $\varphi_{0m} \to \varphi_0$ in H.

Notice that, from Eq. (16) above, we get

$$-\Delta^2\psi_m(t) = \Delta\varphi_m(t) = \sum_{j=1}^{m} g_{jm}(t)\Delta\varphi_j = -\sum_{j=1}^{m} g_{jm}(t)\Delta^2\psi_j$$

and since Δ^2 is an isometry in V_ψ we have

$$\psi_m(t) = \sum_{j=1}^{m} g_{jm}(t)\psi_j.$$

In other words, our definition (14) of the functions $\psi_m(t)$ is compatible with the solution of the discretized equation (16). Therefore, if we use Eq. (15) to determine the functions $g_{jm}(t)$, $j = 1, 2, \ldots, m$ then Eq. (16) will automatically be satisfied. Also, it follows from Eq. (12) that

$$\Delta\psi_m(t) = \varphi_m(t) \quad \text{in } H. \tag{17}$$

Since $(\varphi_i)_{i=1}^m$ is a basis for V_φ, we may write

$$\varphi_{0m} = \sum_{i=1}^{m} g_{im}^0\varphi_i. \tag{18}$$

The system of Eqs. (15) and (16) will be labelled Problem $(\mathrm{P_m})$.

Lemma 9. *If $g_{1m}(t), g_{2m}(t), \ldots, g_{mm}(t)$ are found such that Eq. (16) are satisfied, then*

$$\|\Delta\psi_m(t)\| \leqslant c\|\nabla\varphi_m(t)\|$$

and

$$\|\psi_m(t)\| \leqslant c\|\varphi_m(t)\|.$$

Proof. Multiplying both sides of Eq. (16) by $g_{im}(t)$ and summing from 1 to m we obtain

$$\|\Delta\psi_m(t)\|^2 = -(\nabla\varphi_m(t), \nabla\psi_m(t)) \leqslant \|\nabla\varphi_m(t)\|\|\nabla\psi_m(t)\| \leqslant c\|\nabla\varphi_m(t)\|\|\Delta\psi_m(t)\|.$$

The second statement follows from (17) and the fact that $\|\psi_m(t)\| \leqslant c\|\Delta\psi_m(t)\|$. □

Lemma 10. *Problem* $(\mathrm{P_m})$ *has a unique solution, i.e., there exists a unique set of functions*

$$g_{1m}(t), g_{2m}(t), \ldots, g_{mm}(t) \in L^\infty(0,T)$$

such that the functions $\varphi_m(t)$, $\psi_m(t)$ *defined by* (13) *and* (14) *satisfy* $(\mathrm{P_m})$. *Moreover,*

$$\varphi_m \in L^2(0,T;V_\varphi) \cap L^\infty(0,T;H), \qquad \psi_m \in L^\infty(0,T;V_\psi)$$

with norms bounded uniformly in m.

Proof. Using the definitions (13) and (14) of the functions $\varphi_m(t)$, $\psi_m(t)$ in (15), we see that the functions $g_{jm}(t)$, $j = 1,2,\ldots,m$ must satisfy the system of ordinary differential equations

$$\sum_{j=1}^{m}\langle\varphi_j,\varphi_i\rangle\frac{\mathrm{d}}{\mathrm{d}t}g_{jm}(t) + \nu\sum_{j=1}^{m}(\nabla\varphi_j,\nabla\varphi_i)g_{jm}(t) + \sum_{j=1}^{m}\sum_{k=1}^{m}\langle B(\varphi_j,\psi_k),\varphi_i\rangle g_{jm}(t)g_{km}(t)$$
$$+\sum_{j=1}^{m}\langle B(\varphi_j,\psi_b),\varphi_i\rangle g_{jm}(t)$$
$$= \langle q,\varphi_i\rangle, \quad g_{im}(0) = g^0_{im}, \quad i = 1,2,\ldots,m,$$

which may be written in a matrix form as

$$\frac{\mathrm{d}}{\mathrm{d}t}G_m(t) = \Lambda^{-1}(f(G_m(t),G_m(t)) + f_0), \tag{19}$$
$$G_m(0) = G_0, \tag{20}$$

where

$$G_m(t) = [\,g_{1m}(t) \quad \cdots \quad g_{mm}(t)\,]^{\mathrm{T}},$$
$$\Lambda = [\langle\varphi_j,\varphi_i\rangle],$$
$$G_0 = [\,g^0_{1m} \quad \cdots \quad g^0_{mm}\,]^{\mathrm{T}},$$

$f(\cdot,\cdot)$ is a bilinear function and f_0 is a constant vector. Λ has an inverse because it is the Gramian of a system of linearly independent functions. The system (19) and (20) has a unique solution defined in a maximal interval $t_m \leqslant T$. Correspondingly, we have a unique solution $\varphi_m(t)$ of (15) defined by (13) in $[0,t_m]$. A standard argument (see [4]), which involves multiplying both sides of Eq. (15) by $g_{im}(t)$ and summing from 1 to m, is used to show that $\varphi_m(\cdot) \in L^2(0,T;V_\varphi) \cap L^\infty(0,T;H)$. Eq. (17) gives $\psi_m(\cdot) \in L^\infty(0,T;V_\psi)$. □

We remark that $L^\infty(0,T;V_\psi) \subset L^2(0,T;V_\psi)$. The definition of the space $\mathscr{H}^\gamma(0,T;X,Y)$ for $0<\gamma$ and Hilbert spaces X, Y used in the following lemma can be found in [4].

Lemma 11. *The sequence* $(\varphi_m(\cdot))$ *is bounded in* $\mathscr{H}^\gamma(0,T;V_\varphi,H)$ *for* $0<\gamma<\frac{1}{4}$.

Proof. Denote by $\widetilde{\varphi}_m(\cdot)$, $\widetilde{\psi}_m(\cdot)$, $\widetilde{q}(\cdot)$ the extensions of the functions $\varphi_m(\cdot)$, $\psi_m(\cdot)$, $q(\cdot)$ by zero outside $[0,T]$. Then (15) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\widetilde{\varphi}_m(t),\varphi_i\rangle + \nu(\nabla\widetilde{\varphi}_m(t),\nabla\varphi_i) + \langle B(\widetilde{\varphi}_m(t),\widetilde{\psi}_m(t)),\varphi_i\rangle + \langle B(\widetilde{\varphi}_m(t),\psi_b),\varphi_i\rangle$$
$$= \langle\widetilde{q}(t),\varphi_i\rangle + (\varphi_{0m},\varphi_i)\delta_0 - (\varphi_m(T),\varphi_i)\delta_T, \quad i = 1,2,\ldots,m,$$

where δ_0 and δ_T denote the Dirac delta functions at $t = 0$ and $t = T$, respectively. Denote by $\widehat{\varphi}_m(\tau)$, $\widehat{\psi}_m(\tau)$, $\widehat{B}_m(\tau)$, and $\widehat{q}(\tau)$ the Fourier transforms of $\widetilde{\varphi}_m(t)$, $\widetilde{\psi}_m(t)$, $B(\widetilde{\varphi}_m(t), \widetilde{\psi}_m(t))$, and $\widetilde{q}(t)$ respectively. The Fourier transform of the above system is

$$\begin{aligned}&2\pi i\tau(\widehat{\varphi}_m(\tau), \varphi_j) + v(\nabla\widehat{\varphi}_m(\tau), \nabla\varphi_j) + \langle \widehat{B}_m(\tau), \varphi_j\rangle + \langle B(\widehat{\varphi}_m(\tau), \psi_b), \varphi_i\rangle \\ &\quad = \langle \widehat{q}(\tau), \varphi_i\rangle + (\varphi_{0m}, \varphi_i) - (\varphi_m(T), \varphi_i)\mathrm{e}^{-2\pi i\tau T}, \quad i = 1, 2, \ldots, m.\end{aligned}$$

Hence,

$$2\pi i\tau\|\widehat{\varphi}_m(\tau)\|^2 + v\|\widehat{\varphi}_m(\tau)\|^2_{V_\varphi} + \langle \widehat{B}_m(\tau), \widehat{\varphi}_m(\tau)\rangle = \langle \widehat{q}(\tau), \widehat{\varphi}_m(\tau)\rangle + (\varphi_{0m}, \widehat{\varphi}_m(\tau)) - (\varphi_m(T), \widehat{\varphi}_m(\tau))\mathrm{e}^{-2\pi i\tau T}.$$

The imaginary part of this equality yields

$$2\pi|\tau|\|\widehat{\varphi}_m(\tau)\|^2 \leqslant (\|\widehat{q}(\tau)\|_{V'_\varphi} + \|\widehat{B}_m(\tau)\|_{V'_\varphi})\|\widehat{\varphi}_m(\tau)\|_{V_\varphi} + (\|\varphi_{0m}\| + \|\varphi_m(T)\|)\|\widehat{\varphi}_m(\tau)\|.$$

Now from (8), Lemmas 9 and 10, we get

$$\begin{aligned}&\|\widehat{B}_m(\tau)\|_{V_\varphi} \leqslant \int_{-\infty}^{\infty} \|B(\widetilde{\varphi}_m(t), \widetilde{\psi}_m(t))\|_{V_\varphi}\,\mathrm{d}t \leqslant c\int_0^T \|\nabla\varphi_m(t)\|\|\Delta\psi_m(t)\|\,\mathrm{d}t \leqslant c\int_0^T \|\nabla\varphi_m(t)\|^2\,\mathrm{d}t \leqslant c,\\ &\|\widehat{q}(\tau)\|_{V_\varphi} \leqslant \int_{-\infty}^{\infty} \|\widetilde{q}(t)\|_{V_\varphi}\,\mathrm{d}t \leqslant T^{1/2}\|q\|_{L^2(0,T;V_\varphi)} \leqslant c.\end{aligned}$$

Hence, we have the following bound for all $\tau \in \mathbb{R}$.

$$|\tau|\|\widehat{\varphi}_m(\tau)\|^2 \leqslant c(\|\widehat{\varphi}_m(\tau)\|_{V_\varphi} + \|\widehat{\varphi}_m(\tau)\|).$$

Following an argument similar to [4, p. 165] we get

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma}\|\widehat{\varphi}_m(\tau)\|^2\,\mathrm{d}\tau \leqslant c. \qquad \square$$

Corollary 12. *The sequence* (ψ_m) *is bounded in* $\mathscr{H}^\gamma(0, T; V_\psi, V_\psi)$ *for* $0 < \gamma < \frac{1}{4}$.

Proof. This follows from Lemma 9 and Parseval's identity. $\square$

Theorem 13. *Problem* (P) *has at least one solution pair* (φ, ψ) *where*

$$\varphi(\cdot) \in L^2(0, T; V_\varphi) \cap L^\infty(0, T; H), \qquad \psi(\cdot) \in L^\infty(0, T; V_\psi).$$

Proof. By virtue of Lemmas 10 and 11 and Corollary 12 there exists a subsequence of (φ_m, ψ_m) (we still denote it by (φ_m, ψ_m)) such that

$$\begin{aligned}&(\varphi_m, \psi_m) \rightharpoonup (\varphi, \psi) \quad \text{in } L^2(0, T; V_\varphi) \times L^2(0, T; V_\psi),\\ &(\varphi_m, \psi_m) \overset{*}{\rightharpoonup} (\varphi, \psi) \quad \text{in } L^\infty(0, T; H) \times L^\infty(0, T; V_\psi),\\ &(\varphi_m, \psi_m) \rightharpoonup (\varphi, \psi) \quad \text{in } \mathscr{H}^\gamma(0, T; V_\varphi, H) \times \mathscr{H}^\gamma(0, T; V_\psi, V_\varphi).\end{aligned} \tag{21}$$

Since the compact embedding of a Hilbert space X into a Hilbert space Y implies the compact embedding of $\mathscr{H}^\gamma(0, T; X, Y)$ into $L^2(0, T; Y)$, we have

$$(\varphi_m, \psi_m) \to (\varphi, \psi) \quad \text{in } L^2(0, T; H) \times L^2(0, T; H^1(\Omega)). \tag{22}$$

Without loss of generality, we may assume that the basis functions (φ_i) are in $\mathscr{D}(\Omega)$. Let $\theta \in C^1([0, T])$ be such that $\theta(T) = 0$. The weak convergence in $L^2(0, T; V_\varphi) \times L^2(0, T; V_\psi)$ gives

$$\begin{aligned}&(\nabla\varphi_m(t), \nabla\eta) \to (\nabla\varphi(t), \nabla\eta),\\ &(\Delta\psi_m(t), \Delta\eta) \to (\Delta\psi(t), \Delta\eta).\end{aligned}$$

For almost all $t \in [0, T]$ and all η in V_ψ. Hence,

$$-((\Delta\psi(t), \Delta\eta) + (\nabla\varphi(t), \nabla\eta)) = 0,$$

and Eq. (11) follows. To show the validity of Eq. (10) we first multiply both sides by θ and integrate over $[0, T]$ and use Green's formula; this gives

$$-\int_0^T \langle\varphi_m(t), \varphi_i\rangle\theta'(t)\,\mathrm{d}t + \int_0^T (v(\nabla\varphi_m(t), \nabla\varphi_i) + \langle B(\varphi_m(t), \psi_m(t)), \varphi_i\rangle)\theta(t)\,\mathrm{d}t + \int_0^T \langle B(\varphi_m(t), \psi_b), \varphi_i\rangle\theta(t)\,\mathrm{d}t$$
$$= \int_0^T \langle q(t), \varphi_i\rangle\theta(t)\,\mathrm{d}t + \langle\varphi_{0m}, \varphi_i\rangle\theta(0),$$

$i = 1, 2, \ldots, m$. Hence, for a fixed m_0 and any $\xi \in V_\varphi^{m_0}$, we have

$$-\int_0^T (\varphi_m(t), \xi)\theta'(t)\,\mathrm{d}t + \int_0^T (v(\nabla\varphi_m(t), \nabla\xi) + \langle B(\varphi_m(t), \psi_m(t)), \xi\rangle)\theta(t)\,\mathrm{d}t + \int_0^T \langle B(\varphi_m(t), \psi_b), \xi\rangle\theta(t)\,\mathrm{d}t$$
$$= \int_0^T \langle q(t), \xi\rangle\theta(t)\,\mathrm{d}t + \langle\varphi_{0m}, \xi\rangle\theta(0).$$

It follows from (21) that

$$\int_0^T (\varphi_m(t), \xi)\theta'(t)\,\mathrm{d}t \to \int_0^T (\varphi(t), \xi)\theta'(t)\,\mathrm{d}t,$$
$$\int_0^T (\nabla\varphi_m(t), \nabla\xi)\theta(t)\,\mathrm{d}t \to \int_0^T (\nabla\varphi(t), \nabla\xi)\theta(t)\,\mathrm{d}t.$$

Furthermore, Lemma 3 and the weak convergence

$$\varphi_m \rightharpoonup \varphi \quad \text{in } L^2(0, T; V_\varphi)$$

give

$$\int_0^T \langle B(\varphi_m(t), \psi_b), \xi\rangle\theta(t)\,\mathrm{d}t = -\int_0^T \langle B(\xi, \psi_b), \varphi_m(t)\rangle\theta(t)\,\mathrm{d}t \to -\int_0^T \langle B(\xi, \psi_b), \varphi(t)\rangle\theta(t)\,\mathrm{d}t$$
$$= \int_0^T \langle B(\varphi(t), \psi_b), \xi\rangle\theta(t)\,\mathrm{d}t.$$

Next using (22) we show that

$$\int_0^T \langle B(\varphi_m(t), \psi_m(t)), \xi\rangle\theta(t)\,\mathrm{d}t \to \int_0^T \langle B(\varphi(t), \psi(t)), \xi\rangle\theta(t)\,\mathrm{d}t.$$

We have

$$\left|\int_0^T \langle B(\varphi_m(t), \psi_m(t)), \xi\rangle\theta(t)\,\mathrm{d}t - \int_0^T \langle B(\varphi(t), \psi(t)), \xi\rangle\theta(t)\,\mathrm{d}t\right|$$
$$= \left|\int_0^T (\langle B(\varphi_m(t) - \varphi(t), \psi_m(t)), \xi\rangle + \langle B(\varphi(t), \psi_m(t) - \psi(t)), \xi\rangle)\theta(t)\,\mathrm{d}t\right|$$
$$= \left|-\int_0^T (\langle B(\xi, \psi_m(t)), \varphi_m(t) - \varphi(t)\rangle + \langle B(\xi, \psi_m(t) - \psi(t)), \varphi(t)\rangle)\theta(t)\,\mathrm{d}t\right|$$
$$\leqslant \|\nabla\xi\|_\infty \int_0^T (\|\nabla\psi_m(t)\|\|\varphi_m(t) - \varphi(t)\| + \|\nabla\psi_m(t) - \nabla\psi(t)\|\|\varphi(t)\|)|\theta(t)|\,\mathrm{d}t$$
$$\leqslant \|\nabla\xi\|_\infty\|\theta\|_\infty\left(\left[\int_0^T \|\nabla\psi_m(t)\|^2\right]^{1/2}\left[\int_0^T \|\varphi_m(t) - \varphi(t)\|^2\right]^{1/2}\mathrm{d}t\right.$$
$$\left. + \left[\int_0^T \|\varphi(t)\|^2\right]^{1/2}\left[\int_0^T \|\nabla\psi_m(t) - \nabla\psi(t)\|^2\right]^{1/2}\mathrm{d}t\right).$$

The right hand side of the last inequality converges to 0 since $\varphi_m(\cdot) \to \varphi(\cdot)$ strongly in $L^2(0,T;H)$ and $\psi_m(\cdot) \to \psi(\cdot)$ strongly in $L^2(0,T;H^1(\Omega))$. Finally, by hypothesis, $\varphi_{0m} \to \varphi_0$ in H. Hence, in the limit, we have

$$-\int_0^T \langle \varphi(t), \xi\rangle \theta'(t)\,dt + \int_0^T (v(\nabla\varphi(t), \nabla\xi) + \langle B(\varphi(t), \psi(t)), \xi\rangle)\theta(t)\,dt + \int_0^T \langle B(\varphi(t), \psi_b), \xi\rangle\theta(t)\,dt$$
$$= \int_0^T \langle q(t), \xi\rangle\theta(t)\,dt + \langle \varphi_0, \xi\rangle\theta(0), \tag{23}$$

for all $\theta \in C^1([0,T])$ with $\theta(T) = 0$. Since m_0 is arbitrary and $\cup_{m\geqslant 1} V_\varphi^m$ is dense in V_φ it follows that (23) is also valid for all $\xi \in V_\varphi$. The first of Eq. (10) now follows by restricting θ to $\mathscr{D}(\Omega)$.

It remains to check that $\varphi(0) = \varphi_0$. This can be shown by multiplying both sides of the first equation in (10) by a function $\theta \in C^1([0,T])$ with $\theta(T) = 0$, integrating from 0 to T, using Green's formula and comparing with (23). We get $\langle\varphi(0), \xi\rangle = \langle\varphi_0, \xi\rangle$, $\forall\xi \in V_\varphi$. Hence, $\varphi(0) = \varphi_0$ in V_φ, and consequently, $\varphi(0) = \varphi_0$ in H, since $\varphi_0 \in H$. □

Theorem 14. *Problem* (P) *has a unique solution* (φ, ψ) *where*

$$\varphi(\cdot) \in L^2(0,T;V_\varphi) \cap L^\infty(0,T;H), \qquad \psi(\cdot) \in L^\infty(0,T;V_\psi).$$

Proof. Suppose that $(\varphi_1(\cdot), \psi_1(\cdot))$, $(\varphi_2(\cdot), \psi_2(\cdot))$ are solutions. Let $\zeta(\cdot) = \varphi_1(\cdot) - \varphi_2(\cdot)$, $\varsigma(\cdot) = \psi_1(\cdot) - \psi_2(\cdot)$ then for all $\chi \in V_\varphi$ and all $\xi \in V_\psi$,

$$\begin{aligned} &\frac{d}{dt}\langle\zeta(t), \chi\rangle + v(\nabla\zeta(t), \nabla\chi) + \langle B(\varphi_1(t), \psi_1(t)), \chi\rangle - \langle B(\varphi_2(t), \psi_2(t)), \chi\rangle + \langle B(\zeta(t), \psi_b), \chi\rangle = 0, \\ &(\Delta\varsigma(t), \Delta\xi) + (\nabla\zeta(t), \nabla\xi) = 0. \end{aligned} \tag{24}$$

The second equation above gives $\Delta^2\varsigma(t) = \Delta\zeta(t)$ or $\Delta\varsigma(t) = \zeta(t)$ in H.

If in (24) above we use $\chi = \zeta(t)$, $\xi = \varsigma(t)$ then

$$\left\langle \frac{d}{dt}\zeta(t), \zeta(t) \right\rangle + v\|\nabla\zeta(t)\|^2 + \langle B(\varphi_1(t), \psi_1(t)), \zeta(t)\rangle - \langle B(\varphi_2(t), \psi_2(t)), \zeta(t)\rangle = 0.$$

Using the estimates from Lemma 4 and since

$$\begin{aligned} &\langle B(\varphi_1(t), \psi_1(t)), \zeta(t)\rangle - \langle B(\varphi_2(t), \psi_2(t)), \zeta(t)\rangle = \langle B(\zeta(t), \varsigma(t)), \varphi_2(t)\rangle, \\ &|\langle B(\varphi_1(t), \psi_1(t)), \zeta(t)\rangle - \langle B(\varphi_2(t), \psi_2(t)), \zeta(t)\rangle| \leqslant c\|\nabla\zeta(t)\|\|\Delta\varsigma(t)\|\|\nabla\varphi_2(t)\| \\ &\quad = c\|\nabla\varphi_2(t)\|\|\nabla\zeta(t)\|\|\zeta(t)\| \leqslant \frac{v}{2}\|\nabla\zeta(t)\|^2 + \frac{c}{2v}\|\nabla\varphi_2(t)\|^2\|\zeta(t)\|^2. \end{aligned}$$

Hence,

$$\frac{d}{dt}\|\zeta(t)\|^2 \leqslant \frac{c}{2v}\|\nabla\varphi_2(t)\|^2\|\zeta(t)\|^2.$$

Since $\|\zeta(0)\|^2 = \|\varphi_1(0) - \varphi_2(0)\|^2 = \|\varphi_0 - \varphi_0\|^2 = 0$, then, by Gronwal's inequality, $\|\zeta\|^2 = 0$ or $\zeta \equiv 0$. Then

$$\Delta\varsigma = 0.$$

Since ς satisfies homogeneous boundary conditions we get $\varsigma \equiv 0$. □

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