Pergamon

Computers Math. Applic. Vol. 36, No. 2, pp. 117–127, 1998

0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00123-0

A Two-Level Finite-Element Discretization of the Stream Function Form of the Navier-Stokes Equations

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(Received August 1997; accepted October 1997)

Abstract—We analyze a two-level method of discretizing the stream function form of the Navier-Stokes equations. This report presents the two-level algorithm and error analysis for the case of conforming elements. The two-level algorithm consists of solving a small nonlinear system on the coarse mesh, then solving a linear system on the fine mesh. The basic result states that the error between the coarse and fine meshes are related superlinearly via:

$$\left|\psi - \psi^h\right|_2 \le C\left\{\inf_{w^h\in X^h}\left|\psi - w^h\right|_2 + |\ln h|^{1/2}\cdot\left|\psi - \psi^H\right|_1\right\}.$$

As an example, if the Clough-Tocher triangles or the Bogner-Fox-Schmit rectangles are used, then the coarse and fine meshes are related by $h = O(H^{3/2}|\ln H|^{1/4})$. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Navier-Stokes equations, Reynolds number, Finite element, Two-level methods, Stream function formulation.

1. INTRODUCTION

Convergence analysis for finite-element approximation of the primitive variable formulation of the Navier-Stokes equations have been extensively developed in the last 20 years, see, for example, [1–4]. The analogous theory for the stream function formulation for the Navier-Stokes equations has received much less attention. The attractions of the stream function formulation are that the incompressibility constraint is automatically satisfied, the pressure is not present in the weak form, and there is only one scalar unknown to solve for. The standard weak formulation of the stream function version first appeared in 1979 in [2]. In this direction, Cayco and Nicolaides [5,6] studied a general analysis of convergence for this standard weak formulation of the Navier-Stokes equations. The standard weak form is unsuitable for derivation or analysis of nonconforming finite-element approximations. For a nonconforming finite-element method, Baker and Jureidini [7] investigated the use of elements which are required only to be continuous and are not required to satisfy the boundary conditions with a nonstandard weak formulation. Their weak formulation extends the standard one by including appropriate integrals on interelement boundaries and on the boundary of the problem domain. Cayco and Nicolaides [6] presented

I would like to thank W. Layton for introducing me to the subject and for the helpful discussion related to this work. I would also like to thank L. Tobiska for giving valuable comments related to this work.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX and discussed a new weak form, which is suitable for analysis of nonconforming finite-element approximations. They discussed this weak form and applied it to three specific nonconforming finite-element schemes.

The discretization of the stream function formulation still leads to a problem of solving a large and ill-conditioned nonlinear systems of algebraic equations. Two-level finite-element discretizations are presently a very promising approach for approximating the Navier-Stokes equations, see [8]. The computational attractions of the methods are that they require the solution of only a small system of nonlinear equations on coarse mesh and one linear system of equations on fine mesh. These types of methods were pioneered by Xu in [9,10] for semilinear elliptic problems. The two-level discretization methods have been recently analyzed for the Navier-Stokes equations in [8,11,12] and for the stream function formulation of the Navier-Stokes equations in [13]. The methods studied in [13] involve solving a full linearization of the stream function equation on the fine mesh. The purpose of this paper is to present and analyze a two-level conforming finite-element method of discretizing the stream function formulation of the Navier-Stokes equations which requires the solution of a partial linearization of the stream function equation on the fine mesh. The use of partial linearization of the stream function equation on the fine mesh is due to the possible indefinite matrix resulting from the full linearization. While the partial linearization results are due to a positive definite matrix.

2. NOTATION AND PRELIMINARIES

We first need to define some function spaces and associated norms. More details concerning these spaces can be found in [14]. Let Ω be a bounded, simply connected, polygonal domain in R^2. $L^2(\Omega)$ is the Hilbert space of Lebesgue square integrable functions with norm $\|\cdot\|_0$ and $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions with zero mean. Let $H^m(\Omega)$ be the usual Sobolev space consisting of functions, which together with their distributional derivatives up through order m are in $L^2(\Omega)$. Denote the norm on $H^m(\Omega)$ by $\|\cdot\|_m$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the $\|\cdot\|_m$ norm. We equip $H_0^m(\Omega)$ with the seminorm $|\cdot\|_m$, which is a norm equivalent to $\|\cdot\|_m$. Also, the dual of space $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$, with norm $\|\cdot\|_{-m}$. Let $[H^m(\Omega)]^2$ be the space $H^m(\Omega)\times H^m(\Omega)$ and $[H_0^m(\Omega)]^2$ be the space $H_0^m(\Omega)\times H_0^m(\Omega)$ equipped with the following norm:

$$\|\vec{u}\|_m = \left(\|u_1\|_m^2 + \|u_2\|_m^2\right)^{1/2} \text{ and } |\vec{u}|_m = \left(|u_1|_m^2 + |u_2|_m^2\right)^{1/2}, \qquad \text{where } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

For each $\phi \in H^1(\Omega)$, define

$$\vec{\operatorname{curl}}\,\phi = \begin{pmatrix} \phi_y \\ -\phi_x \end{pmatrix}.$$

For each $\vec{u} \in [H^1(\Omega)]^2$, define

$$\operatorname{curl}\vec{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \qquad \text{where } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Consider the Navier-Stokes equations describing the flow of an incompressible fluid:

$$\begin{aligned} -\operatorname{Re}^{-1}\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p &= \vec{f}, && \text{in } \Omega, \\ \nabla\cdot\vec{u} &= 0, && \text{in } \Omega, \\ \vec{u} &= 0, && \text{on } \partial\Omega, \\ \int_\Omega p\, d\Omega &= 0. && \end{aligned} \tag{1}$$

Later, we will state conditions on $\vec{f}$ and Re^{-1} guaranteeing the solution to (1). Any divergence-free velocity vector $\vec{u} \in [H_0^1(\Omega)]^2$ has a unique stream function [2, Theorem 3.1, p. 22] $\psi \in H_0^2(\Omega)$, defined by

$$\vec{\mathrm{curl}}\, \psi = \vec{u}.$$

Moreover, the stream function ψ satisfies

$$\begin{aligned} \mathrm{Re}^{-1} \Delta^2 \psi - \psi_y \Delta \psi_x + \psi_x \Delta \psi_y &= \vec{\mathrm{curl}}\, \vec{f}, && \text{in } \Omega, \\ \psi &= 0, && \text{on } \partial\Omega, \\ \frac{\partial \psi}{\partial \hat{n}} &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\hat{n}$ represents the outward unit normal to Ω.

3. TWO WEAK FORMULATIONS

The standard weak form of equation (1) is:

$$\begin{gathered} \text{find } \vec{u} \in \left[H_0^1\right]^2,\ p \in L_0^2(\Omega), \text{ such that } \forall\, \vec{w} \in \left[H_0^1(\Omega)\right]^2,\ q \in L_0^2(\Omega), \\ \mathrm{Re}^{-1}\, \tilde{a}(\vec{u}, \vec{w}) + \tilde{b}(\vec{u}; \vec{u}, \vec{w}) + \tilde{c}(\vec{w}, p) = \left\langle \vec{f}, \vec{w} \right\rangle,\ \tilde{c}(\vec{u}, q) = 0, \end{gathered} \tag{3}$$

where

$$\begin{aligned} \tilde{a}(\vec{u}, \vec{w}) &= \int_\Omega \nabla \vec{u} : \nabla \vec{w}, \\ \tilde{b}(\vec{u}; \vec{v}, \vec{w}) &= \int_\Omega ((\vec{u} \cdot \nabla)\, \vec{v}) \cdot \vec{w}, \\ \tilde{c}(\vec{w}, q) &= \int_\Omega q \operatorname{div} \vec{w}, \end{aligned} \tag{4}$$

and $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $L^2(\Omega)$. The standard weak form of equation (2) is:

$$\text{find } \psi \in H_0^2(\Omega) \text{ such that, for all } \phi \in H_0^2(\Omega), \qquad a(\psi, \phi) + b(\psi; \psi, \phi) = l(\phi), \tag{5}$$

where

$$\begin{aligned} a(\psi, \phi) &= \mathrm{Re}^{-1} \int_\Omega \Delta \psi \cdot \Delta \phi, \\ b(\xi; \psi, \phi) &= \int_\Omega \Delta \xi (\psi_y \phi_x - \psi_x \phi_y), \\ l(\phi) = \left(\vec{f}, \vec{\mathrm{curl}}\, \phi\right) &= \int_\Omega \vec{f} \cdot \vec{\mathrm{curl}}\, \phi. \end{aligned} \tag{6}$$

Another equivalent formulation of equation (2), introduced by Cayco and Nicolaides [6], is:

$$\begin{gathered} \text{find } \psi \in H_0^2(\Omega) \text{ such that, for all } \phi \in H_0^2(\Omega), \\ a_0(\psi, \phi) + b_0(\psi; \psi, \phi) = l(\phi), \end{gathered} \tag{7}$$

where

$$\begin{aligned} a_0(\psi, \phi) &= \mathrm{Re}^{-1} \int_\Omega \psi_{xx}\phi_{xx} + 2\psi_{xy}\phi_{xy} + \psi_{yy}\phi_{yy}, \\ b_0(\xi; \psi, \phi) &= \int_\Omega (\xi_y \psi_{xy} - \xi_x \psi_{yy})\phi_y - (\xi_x \psi_{xy} - \xi_y \psi_{xx})\phi_x, \\ l(\phi) = \left(\vec{f}, \vec{\mathrm{curl}}\, \phi\right) &= \int_\Omega \vec{f} \cdot \vec{\mathrm{curl}}\, \phi. \end{aligned} \tag{8}$$

Conforming elements can be used with either (5) or (7); in this case, the two weak formulations produce identical results because $a(\psi, \phi) = a_0(\psi, \phi)$ and $b(\xi; \psi, \phi) = b_0(\xi; \psi, \phi)$, for all

$\psi, \phi, \xi \in H_0^2(\Omega)$. However, when using nonconforming approximating subspaces, (7) and (5) generate different finite-element methods. Nonconforming elements should be used only with (7). To illustrate the reason, suppose we solve the Stokes problem with the nonconforming Morley triangle, i.e., the quadratic element whose degrees of freedom are function values at the vertices and normal derivatives at the midsides. Boundary conditions are imposed by setting all the degrees of freedom at the boundary to be zero. Observe that a necessary and sufficient condition for the existence of a unique solution to the discrete biharmonic equation is that the bilinear induces a norm on the trial space. This is not the case for the Morley space [5]. The following theorem states that the forms (3) and (5) are equivalent in the sense of having identical solutions. The reason for this is that the space of curls of $H_0^2(\Omega)$ functions coincides with the space of divergence-free functions in $[H_0^1(\Omega)]^2$.

The following theorem states that problems (1) and (2) are equivalent in the sense of having identical solutions.

THEOREM 3.1. *(See [2, Theorem 2.6, p. 120].) Problems (3) and (5) are equivalent in the sense that if $(\vec{u}, p)$ is a solution of (3), then the stream-function ψ of $\vec{u}$ satisfies (5); conversely, if ψ is a solution of (5), then there exists exactly one element p of $L_0^2(\Omega)$ such that the pair $(\vec{u} = \vec{\mathrm{curl}}\, \psi, p)$ satisfies (3).*

The following lemma states some basic bound for the bilinear a, the trilinear b, and the functional l.

LEMMA 3.1. *Given $\psi, \xi, \phi \in H_0^2(\Omega)$ and $\vec{f} \in [L^2(\Omega)]^2$, there exists a $C > 0$ such that*

$$a(\psi, \psi) = \mathrm{Re}^{-1}\, |\psi|_2^2, \tag{9}$$

$$a(\psi, \phi) \le \mathrm{Re}^{-1}\, |\psi|_2 \cdot |\phi|_2, \tag{10}$$

$$|b(\xi, \psi, \phi)| \le 2C_s^2 |\xi|_2 \cdot |\psi|_2 \cdot |\phi|_2, \tag{11}$$

$$|b_0(\xi, \psi, \phi)| \le C |\xi|_2 \cdot |\psi|_2 \cdot |\phi|_2, \tag{12}$$

$$\left|\left(\vec{f}, \vec{\mathrm{curl}}\, \phi\right)\right| \le \left|\vec{f}\right|_* \cdot |\phi|_2, \tag{13}$$

$$\left|\left(\vec{f}, \vec{\mathrm{curl}}\, \phi\right)\right| \le C_p \left\|\vec{f}\right\|_0 \cdot |\phi|_2, \tag{14}$$

where C_s is a Sobolev embedding constant and C_p is a Poincaré constant.

PROOF. For $\psi, \xi, \phi \in H_0^2(\Omega)$, we have by direct computation, equations (10)–(14). Our task is now to prove (9). We have, by definition,

$$a(\psi, \psi) = |\Delta\, \psi|_0^2 = \int_\Omega \left\{ \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 + 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} \right\}, \tag{15}$$

$$|\psi|_2 = \int_\Omega \left\{ \left(\frac{\partial^2 \psi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \psi}{\partial y^2}\right)^2 + \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 \psi}{\partial y \partial x}\right)^2 \right\}. \tag{16}$$

Clearly, it suffices to prove (9) with $\psi \in \mathcal{D}(\Omega)$; for such a function,

$$\int_\Omega \left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 = -\int_\Omega \frac{\partial \psi}{\partial x} \cdot \frac{\partial^3 \psi}{\partial x \partial y^2} - \int_\Omega \frac{\partial^2 \psi}{\partial x^2} \cdot \frac{\partial^2 \psi}{\partial y^2}, \tag{17}$$

as a double application of Green's formula, and thus (9) is proved. ∎

Let N denote the finite constant

$$N := \sup_{\xi, \psi, \phi \in H_0^2(\Omega)} \frac{|b(\xi, \psi, \phi)|}{|\xi|_2\, |\psi|_2\, |\phi|_2},$$

and $|f|_*$ denote the dual norm:

$$|f|_* := \sup_{\phi \in H_0^2(\Omega)} \frac{(f, \mathrm{curl}\, \phi)}{|\phi|_2}.$$

Then we have the following theorem that can be proved using the method of [2].

THEOREM 3.2. *(See [2].) For $N|f|_* \mathrm{Re}^2 < 1$ and $\vec{f} \in [H^{-1}(\Omega)]^2$, problem (5) has a unique solution ψ. Moreover, there is a unique $p \in L_0^2(\Omega)$ such that $(\vec{\mathrm{curl}}\,\psi, p)$ solves problem (3).*

To study (5) when the uniqueness condition $N|f|_* \mathrm{Re}^2 < 1$ is not valid, then we need to introduce the concept of a nonsingular solution of (2).

DEFINITION 3.1. *Let X and Y be two Banach spaces, F a differentiable mapping from X into Y, F' its derivative, and let $\psi \in X$ be a solution of the equation $F(\psi) = 0$. We say that ψ is a nonsingular solution if there exists a constant $\gamma > 0$ such that*

$$\|F'(\psi) \cdot \phi\|_{Y'} \geq \gamma \|\phi\|_X, \qquad \forall \phi \in X.$$

In the stream function equation case, the mapping $F : H_0^2(\Omega) \longrightarrow [H_0^2(\Omega)]'$ is defined by

$$\langle F(\psi), \phi \rangle = a(\psi, \phi) + b(\psi, \psi, \phi) - (f, \mathrm{curl}\,\phi), \qquad \forall \phi \in H_0^2(\Omega).$$

The nonlinear map F is quadratic and can be shown to be everywhere differentiable in $H_0^2(\Omega)$ and its derivative $F'(\psi) \in \mathcal{L}(H_0^2(\Omega), [H_0^2(\Omega)]')$ is given by

$$\langle F'(\psi) \cdot \phi, \xi \rangle = a(\phi, \xi) + b(\psi, \phi, \xi) + b(\phi, \psi, \xi).$$

Hence, $\psi \in H_0^2(\Omega)$ is a nonsingular solution of (2), if and only if there exists a constant $\gamma > 0$ such that

$$\sup_{\phi \in H_0^2(\Omega)} \frac{a(\xi, \phi) + b(\psi, \xi, \phi) + b(\xi, \psi, \phi)}{|\phi|_2} \geqslant \gamma |\xi|_2, \qquad \forall \xi \in H_0^2(\Omega).$$

4. TWO-LEVEL METHOD

We consider the approximate solution of (2) by a two-level, finite-element procedure. Let X^h, $X^H \subseteq H_0^2(\Omega)$ denote two conforming finite-element meshes with $H \gg h$. The method we consider computes an approximate solution ψ^h in the finite-element space X^h by solving one linear system for the degrees of freedom in X^h. This particular linear problem requires the construction of a finite-element space X^H upon a very coarse mesh of width '$H \gg h$', and then the solution of a much smaller system of nonlinear equations for an approximation in X^H. The solution procedure is then given as follows.

ALGORITHM 4.1.

Step 1. Solve the nonlinear system on coarse mesh for $\psi^H \in X^H$:

$$a\left(\psi^H, \phi^H\right) + b\left(\psi^H, \psi^H, \phi^H\right) = \left(\vec{f}, \vec{\mathrm{curl}}\,\phi^H\right), \qquad \text{for all } \phi^H \in X^H. \tag{18}$$

Step 2. Solve the linear system on fine mesh for $\psi^h \in X^h$:

$$a\left(\psi^h, \phi^h\right) + b\left(\psi^H, \psi^h, \phi^h\right) = \left(\vec{f}, \vec{\mathrm{curl}}\,\phi^h\right), \qquad \text{for all } \phi^h \in X^h. \tag{19}$$

We shall give some examples of finite-element spaces for the stream function formulation. We will impose boundary conditions by setting all the degrees of freedom at the boundary nodes to be zero and the normal derivative equal to zero at all vertices and nodes on the boundary. The inclusion $X^H \subset H_0^2(\Omega)$ requires the use of finite-element functions that are continuously differentiable over Ω.

ARGYIS TRIANGLE. The functions are quintic polynomials within each triangle and the 21 degrees of freedom are chosen to be the function value, the first and second derivatives at the vertices, and the normal derivative at the midsides.

CLOUGH-TOCHER. Here we subdivide each triangle into three triangles by joining the vertices to the centroid. In each of the smaller triangles, the functions are cubic polynomials. There are then 30 degrees of freedom needed to determine the three different cubic polynomials associated with the three triangles. Eighteen of these are used to ensure that, within the big triangle, the functions are continuously differentiable. The remaining 12 degrees of freedom are chosen to be the function values and the first derivatives at the vertices and the normal derivative at the midsides.

BOGNER-FOX-SCHMIDT RECTANGLE. The functions are bicubic polynomials within each rectangle. The degrees of freedom are chosen to be the function value, the first derivatives, and the mixed second derivative at the vertices. We set the function and the normal derivative values equal to zero at all vertices on the boundary.

BICUBIC SPLINE RECTANGLE. The functions are the product of cubic splines. These functions are bicubic polynomials within each rectangle, are twice continuously differentiable over Ω, and their degrees of freedom are the function values at the nodes (plus some additional ones on the boundary).

Below we prove that ψ^H and ψ^h exist in Steps 1 and 2. Also we will prove that Algorithm 4.1 produces an approximate solution which satisfies the error bound

$$\left|\psi - \psi^h\right|_2 \leq C \left\{ \inf_{w^h \in X^h} \left|\psi - w^h\right|_2 + |\ln h|^{1/2} \cdot \left|\psi - \psi^H\right|_1 \right\}. \tag{20}$$

As an example, consider the case of the Clough-Tocher triangle. For this element (see [1,2,15]) we have the following inequalities:

$$\left|\psi - \psi^h\right|_j \leq C h^{4-j} \qquad (j = 0, 1, 2),$$
$$\left|\psi - \psi^H\right|_j \leq C H^{4-j} \qquad (j = 0, 1, 2).$$

Thus, if we seek an approximate solution ψ^h with the same asymptotic accuracy as ψ^h in $|\cdot|_2$, the above error bound shows that the superlinear scaling, between coarse and fine meshes,

$$h = O\left(H^{3/2} |\ln H|^{1/4}\right) \tag{21}$$

suffices. Analogous scalings between coarse and fine meshes can be calculated from (20) by balancing error terms on the right-hand side of (20) in the same way. For each of the elements described above we give, in Table 1, the scaling between coarse and fine meshes.

Table 1.

Element	$\left\|\psi - \psi^H\right\|_2$	$\left\|\psi - \psi^H\right\|_1$	Scaling
Argyris Triangle	H^4	H^5	$h\|\ln h\|^{-1/4} = O\left(H^{5/2}\right)$
Clough-Tocher Triangle	H^2	H^3	$h\|\ln h\|^{-1/4} = O\left(H^{3/2}\right)$
Bogner-Fox-Schmit Rectangle	H^2	H^3	$h\|\ln h\|^{-1/4} = O\left(H^{3/2}\right)$
Bicubic Spline Rectangle	H^2	H^3	$h\|\ln h\|^{-1/4} = O\left(H^{3/2}\right)$

5. THE ERROR BOUND

The basic bound on $b(,,)$ and $b_0(,,)$, given in Lemma 3.1

$$|b(\psi, \phi, \xi)| \leq N |\psi|_2 \cdot |\phi|_2 \cdot |\xi|_2,$$
$$|b_0(\psi, \phi, \xi)| \leq N |\psi|_2 \cdot |\phi|_2 \cdot |\xi|_2,$$

can be improved. For our purpose, we shall be bounding $|b(\psi, \phi, \xi)|$ with ϕ or ξ in a finite-element space X^h or X^H. Since X^h and X^H are subspaces of X, then they satisfy the following discrete Sobolev inequality: for all $\phi^h \in X^h$ (similarly for X^H),

$$\left\|\nabla \phi^h\right\|_{L^\infty} \leq c |\ln(h)|^{1/2} \left|\phi^h\right|_2.$$

Using the above inequality and Lemma 3.1, we can prove the following lemma.

LEMMA 5.1. *For any $\phi^h \in X^h$, the following inequalities:*

$$\left|b\left(\psi,\phi^h,\xi\right)\right| \le C|\ln(h)|^{1/2}|\psi|_2\cdot|\xi|_1\cdot\left|\phi^h\right|_2,$$
$$\left|b\left(\psi,\xi,\phi^h\right)\right| \le C|\ln(h)|^{1/2}|\psi|_2\cdot|\xi|_1\cdot\left|\phi^h\right|_2,$$

hold.

LEMMA 5.2. *The solution to (18) exists and satisfies $|\psi^H|_2 \le \mathrm{Re}\,|\vec{f}|_*$. Suppose*

$$\mathrm{Re}^2 N\left|\vec{f}\right|_* < 1.$$

Then, the solution ψ^H to (18) is unique.

PROOF. Set $\phi^H = \psi^H$ in (18). This gives

$$\mathrm{Re}^{-1}\left|\psi^H\right|_2^2 = \left(\vec{f},\vec{\mathrm{curl}}\,\psi^H\right) \le |f|_*\left|\psi^H\right|_2,$$

thus $|\psi^H|_2 \le \mathrm{Re}\,|\vec{f}|_*$. This bound implies the existence of the solution to (18) by a compactness argument in X^H. Let ψ_1^H and ψ_2^H be two solutions to (18), and $z^H = \psi_1^H - \psi_2^H$. Then,

$$\begin{aligned}
\mathrm{Re}^{-1}\left|z^H\right|_2^2 &= a\left(z^H,z^H\right) + b\left(\psi_1^H,z^H,z^H\right)\\
&= a\left(\psi_1^H,z^H\right) + b\left(\psi_1^H,\psi_1^H,z^H\right) - \left(a\left(\psi_2^H,z^H\right) + b\left(\psi_1^H,\psi_2^H,z^H\right)\right)\\
&= b\left(\psi_2^H,\psi_2^H,z^H\right) - b\left(\psi_1^H,\psi_2^H,z^H\right)\\
&= -b\left(z^H,\psi_2^H,z^H\right)\\
&\le N\left|z^H\right|_2^2\left|\psi_2^H\right|_2 \le N\,\mathrm{Re}\left|\vec{f}\right|_*\left|z^H\right|_2^2,
\end{aligned}$$

which implies uniqueness of solutions for $(1 - N\,\mathrm{Re}^2\,|\vec{f}|_*) > 0$ as

$$\mathrm{Re}^{-1}\left(1 - N\,\mathrm{Re}^2\left|\vec{f}\right|_*\right)\left|z^H\right|_2^2 \le 0.$$

∎

The next theorem gives the basic error bound after Step 1 in the $|\cdot|_2$-seminorm. Before we state the theorem, we need the following lemma.

LEMMA 5.3. *Let ψ be a nonsingular solution of (2) and provided $|\psi - \psi^H|_2 \le \gamma/2N$, then there is a constant $\gamma^* = \gamma^*(\psi)$ such that*

$$\sup_{\phi\in H_0^2(\Omega)} \frac{a(\xi,\phi) + b\left(\psi^H,\xi,\phi\right) + b(\xi,\psi,\phi)}{|\phi|_2} \geqslant \gamma^*|\xi|_2, \qquad \forall\xi \in H_0^2(\Omega). \tag{22}$$

PROOF. From (3), simply follows that for $|\psi - \psi^H|_2$ small enough (which is the case with $0 < H \le H_0$)

$$\sup_{\phi\in H_0^2(\Omega)}\left\{\frac{a(\xi,\phi) + b\left(\psi^H,\xi,\phi\right) + b(\xi,\psi,\phi)}{|\phi|_2} + \frac{b\left(\psi-\psi^H,\xi,\phi\right)}{|\phi|_2}\right\} \ge \gamma|\xi|_2, \qquad \forall\xi\in H_0^2(\Omega).$$

But it follows from (11) that

$$\sup_{\phi\in H_0^2(\Omega)} \frac{a(\xi,\phi) + b\left(\psi^H,\xi,\phi\right) + b(\xi,\psi,\phi)}{|\phi|_2} + N\left|\psi-\psi^H\right|_2\cdot|\xi|_2 \ge \gamma|\xi|_2, \qquad \forall\xi\in H_0^2(\Omega),$$

or

$$\sup_{\phi\in H_0^2(\Omega)} \frac{a(\xi,\phi) + b\left(\psi^H,\xi,\phi\right) + b(\xi,\psi,\phi)}{|\phi|_2} \ge \left(\gamma - N\left|\psi-\psi^H\right|_2\right)|\xi|_2, \qquad \forall\xi\in H_0^2(\Omega).$$

Hence, we have (3).

∎

THEOREM 5.1.

(a) *If the global uniqueness condition* $\mathrm{Re}^2 N|f|_* < 1$ *holds,* ψ *and* ψ^H *both exist uniquely. The error* $|\psi - \psi^H|_2$ *satisfies*

$$|\psi - \psi^H|_2 \le C(\mathrm{Re}) \inf_{w^H \in X^h} |\psi - w^H|_2,$$

where $C(\mathrm{Re}) = (1 + 2N|f|_* \cdot \mathrm{Re}^2)(1 - N|f|_* \mathrm{Re}^2)^{-1} \le C(\sqrt{N|f|_*})$.

(b) *If the uniqueness condition fails, suppose* ψ *is nonsingular solution of (5). Then, there is an* $H_0 = H_0(\psi, f, \mathrm{Re})$ *and* $c = c(\psi, f, \mathrm{Re}, N)$ *such that for* $H \le H_0$,

$$|\psi - \psi^H|_2 \le c(\psi, f, \mathrm{Re}, N) \inf_{w^H \in X^h} |\psi - w^H|_2, \tag{23}$$

where $c(\psi, f, \mathrm{Re}, N) = \gamma^{-1}(\mathrm{Re}^{-1} + N \cdot \mathrm{Re}\,|f|_*) + 1$.

PROOF. Detailed proof of Part (a) can be found in [5]. It remains to show Part (b). Subtracting (18) from (5), gives the following error equation for (18):

$$a(\psi - \psi^H, \phi^H) + b(\psi, \psi, \phi^H) - b(\psi^H, \psi^H, \phi^H) = 0.$$

Adding the following terms $b(\psi^H, \psi, \phi^H) - b(\psi^H, \psi, \phi^H)$ gives

$$a(\psi - \psi^H, \phi^H) + b(\psi - \psi^H, \psi, \phi^H) + b(\psi^H, \psi - \psi^H, \phi^H) = 0.$$

Let $w^h \in X^h$ be an approximation to ψ in X^h and define $\xi^h = \psi^h - w^h$ and $\eta^h = \psi - w^h$, then the above inequality becomes

$$a(\xi^H, \phi^H) + b(\xi^H, \psi, \phi^H) + b(\psi^H, \xi^H, \phi^H) = a(\eta^H, \phi^H) + b(\eta^H, \psi, \phi^H) + b(\psi^H, \eta^H, \phi^H).$$

Using (22) gives

$$\gamma |\xi^H|_2 \le \sup_{\phi^H \in X^H} \left\{ |\phi^H|_2^{-1} \left(a(\eta^H, \phi^H) + b(\eta^H, \psi, \phi^H) + b(\psi^H, \eta^H, \phi^H) \right) \right\}.$$

In view of (10),(11), we have

$$|\xi^H|_2 \le \gamma^{-1} \left(\mathrm{Re}^{-1} + N \left(|\psi|_2 + |\psi^H|_2 \right) \right) |\eta^H|_2.$$

The triangle inequality ($|\psi - \psi^h|_2 \le |\xi^h|_2 + |\eta^h|_2$) implies (23). ∎

LEMMA 5.4. *Given a solution* ψ^H *to (18), then the solution to the following problem:*

$$\begin{gathered} \text{find } \hat{\psi} \in H_0^2(\Omega) \text{ such that, for all } \phi \in H_0^2(\Omega), \\ a(\hat{\psi}, \phi) + b(\psi^H, \hat{\psi}, \phi) = l(\phi), \end{gathered} \tag{24}$$

exist uniquely and satisfies $\|\hat{\psi}\|_2 \le \mathrm{Re}\,|f|_*$.

PROOF. Introducing the continuous bilinear form $B : H_0^2(\Omega) \times H_0^2(\Omega) \to R$ given by

$$B(\psi, \phi) = a(\psi, \phi) + b(\psi^H, \psi, \phi),$$

B is continuous and coercive. Hence, $\hat{\psi}$ exists uniquely. Setting $\phi = \hat{\psi}$ in (24) implies that

$$\begin{aligned} \mathrm{Re}^{-1} \left\| \hat{\psi} \right\|_2^2 &= l(\hat{\psi}), \\ \left\| \hat{\psi} \right\|_2 &= \mathrm{Re} \frac{l(\hat{\psi})}{\left\| \hat{\psi} \right\|_2} \\ &\le \mathrm{Re} \sup_{\phi \in H_0^2(\Omega)} \frac{l(\phi)}{\|\phi\|_2} \\ &= \mathrm{Re}\,|f|_*. \end{aligned}$$

∎

LEMMA 5.5. *Given a solution ψ^H to (18), then the solution to (19) exists uniquely and satisfies*

$$\|\psi\|_2 \le \mathrm{Re}\,|f|_*.$$

PROOF. The bilinear form B is continuous and coercive on X^h. Hence, ψ exists uniquely. Setting $\phi^h = \psi$ in (19) implies that

$$\begin{aligned}
\mathrm{Re}^{-1}\|\psi\|_2^2 &= l(\psi) \\
&= \mathrm{Re}\,\frac{l(\psi)}{\|\psi\|_2} \\
&\le \mathrm{Re} \sup_{\phi\in H_0^2(\Omega)} \frac{l(\phi)}{\|\phi\|_2} \\
&= \mathrm{Re}\,|f|_*.
\end{aligned}$$

∎

By Green's formula, we obtain the following lemma.

LEMMA 5.6. For $\psi, \xi, \phi \in H_0^2(\Omega)$, we have

$$b(\psi,\xi,\phi) = b_0(\xi,\phi,\psi) - b_0(\phi,\xi,\psi). \tag{25}$$

PROOF. Applying Green's formula to the left-hand side of (25) gives

$$\begin{aligned}
b(\psi,\xi,\phi) &= \int_\Omega \Delta\psi\,(\xi_y\phi_x - \xi_x\phi_y)\,d\Omega \\
&= -\int_\Omega \psi_x\,(\xi_y\phi_x - \xi_x\phi_y)_x + \psi_y\,(\xi_y\phi_x - \xi_x\phi_y)_y \\
&\quad + \int_{\partial\Omega} \frac{\partial\psi}{\partial n}\cdot(\xi_y\phi_x - \xi_x\phi_y) \\
&= \int_\Omega (\xi_{xx}\phi_y + \xi_x\phi_{yx} - \xi_{yx}\phi_x - \xi_y\phi_{xx})\,\psi_x \\
&\quad - \int_\Omega (\xi_{yy}\phi_x + \xi_y\phi_{xy} - \xi_{xy}\phi_y - \xi_x\phi_{yy})\,\psi_y \\
&= \int_\Omega (\xi_{xy}\phi_y + \xi_x\phi_{yy} - \xi_{yy}\phi_x - \phi_{xy}\xi_y)\,\psi_y \\
&\quad - \int_\Omega (\xi_y\phi_{xx} + \phi_x\xi_{yx} - \phi_y\xi_{xx} - \xi_x\phi_{yx})\,\psi_x \\
&= \int_\Omega (\phi_y\xi_{xy} - \phi_x\xi_{yy})\psi_y - (\phi_x\xi_{xy} - \phi_y\xi_{xx})\,\psi_x \\
&\quad - \int_\Omega (\xi_y\phi_{xy} - \xi_x\phi_{yy})\,\psi_y - (\xi_x\phi_{yx} - \xi_y\phi_{xx})\,\psi_x \\
&= b_0(\xi,\phi,\psi) - b_0(\phi,\xi,\psi).
\end{aligned}$$

∎

The main result of this paper is the following theorem. It gives the error bound after Step 2.

THEOREM 5.2. *Let $X^{h,H} \in H_0^2(\Omega)$ be two finite-element spaces. Let ψ be the solution to (2) and ψ^h the solution to (19). Then ψ^h satisfies*

$$|\psi - \psi^h|_2 \le C_1 \inf_{w^h\in X^h} \left|\psi^h - w^h\right|_2 + C_2\sqrt{|\ln h|}\,\left|\psi - \psi^H\right|_1,$$

where $C_1 = 2 + N|f|_*\,\mathrm{Re}^2$ and $C_2 = 2N\cdot\mathrm{Re}^2\,|f|_* c$.

PROOF. Subtracting (19) from (2) yields

$$a\left(\psi - \psi^h, \phi^h\right) + b\left(\psi,\psi,\phi\right) - b\left(\psi^H,\psi^h,\phi^h\right) = 0, \qquad \forall\,\phi^h \in X^h.$$

Using Lemma 5.6 gives

$$
\begin{aligned}
&a\left(\psi-\psi^h,\phi^h\right)+b_0\left(\psi,\phi^h,\psi\right)-b_0\left(\phi^h,\psi,\psi\right)\\
&\quad -b_0\left(\psi^h,\phi^h,\psi^H\right)+b_0\left(\phi^h,\psi^h,\psi^H\right)=0, \qquad \forall\,\phi^h\in X^h.
\end{aligned}
$$

Adding the following terms:

$$
-b_0\left(\psi^h,\phi^h,\psi\right)+b_0\left(\phi^h,\psi^h,\psi\right)+b_0\left(\psi^h,\phi^h,\psi\right)-b_0\left(\phi^h,\psi^h,\psi\right)
$$

gives

$$
\begin{aligned}
&a\left(\psi-\psi^h,\phi^h\right)+b_0\left(\psi-\psi^h,\phi^h,\psi\right)+b_0\left(\phi^h,\psi^h-\psi,\psi\right)\\
&\quad +b_0\left(\psi^h,\phi^h,\psi-\psi^H\right)+b_0\left(\phi^h,\psi^h,\psi^H-\psi\right)=0.
\end{aligned}
$$

Let $w^h\in X^h$ be an approximation to ψ in X^h and define $\xi^h=\psi^h-w^h$ and $\eta^h=\psi-w^h$, then the above inequality becomes

$$
\begin{aligned}
&a\left(\eta^h,\phi^h\right)+b_0\left(\eta^h,\phi^h,\psi\right)-b_0\left(\phi^h,\eta^h,\psi\right)\\
&\quad +b_0\left(\psi^h,\phi^h,\psi-\psi^H\right)+b_0\left(\phi^h,\psi^h,\psi^H-\psi\right)\\
&\quad =a\left(\xi^h,\phi^h\right)+b_0\left(\xi^h,\phi^h,\psi\right)-b_0\left(\phi^h,\xi^h,\psi\right).
\end{aligned}
$$

Setting $\phi^h=\xi^h$ implies

$$
\begin{aligned}
a\left(\xi^h,\xi^h\right)&=a\left(\eta^h,\xi^h\right)+b_0\left(\eta^h,\xi^h,\psi\right)-b_0\left(\xi^h,\eta^h,\psi\right)\\
&\quad +b_0\left(\psi^h,\xi^h,\psi-\psi^H\right)+b_0\left(\xi^h,\psi^h,\psi^H-\psi\right).
\end{aligned}
$$

Using Lemma 5.6 gives

$$
\begin{aligned}
a\left(\xi^h,\xi^h\right)&=a\left(\eta^h,\xi^h\right)+b\left(\psi,\eta^h,\xi^h\right)\\
&\quad +b_0\left(\psi^h,\xi^h,\psi-\psi^H\right)+b_0\left(\xi^h,\psi^h,\psi^H-\psi\right).
\end{aligned}
$$

We will bound the right-hand side of the above inequality as follows:

$$
\begin{aligned}
a\left(\eta^h,\xi^h\right)&\le \mathrm{Re}^{-1}\left|\eta^h\right|_2\cdot\left|\xi^h\right|_2,\\
b\left(\psi,\eta^h,\xi^h\right)&\le N|\psi|_2\cdot\left|\eta^h\right|_2\cdot\left|\xi^h\right|_2,\\
b_0\left(\psi^h,\xi^h,\psi-\psi^H\right)&\le N\cdot c|\psi^h|_2\cdot\left|\xi^h\right|_2\cdot\left|\psi-\psi^H\right|_1\cdot\sqrt{|\ln h|},\\
b_0\left(\xi^h,\psi^h,\psi^H-\psi\right)&\le N\cdot c|\psi^h|_2\cdot\left|\xi^h\right|_2\cdot\left|\psi-\psi^H\right|_1\cdot\sqrt{|\ln h|}.
\end{aligned}
$$

Using these bounds gives

$$
\begin{aligned}
\mathrm{Re}^{-1}\left|\xi^h\right|_2^2&\le \mathrm{Re}^{-1}\left(1+N|\psi|_2\cdot\mathrm{Re}\right)\left|\eta^h\right|_2\cdot\left|\xi^h\right|_2\\
&\quad +2Nc\left|\psi^h\right|_2\cdot\left|\xi^h\right|_2\cdot\left|\psi-\psi^H\right|_1\cdot\sqrt{|\ln h|}.
\end{aligned}
$$

Using the bounds on $|\psi|_2$ and $|\psi^h|_2$ gives

$$
\left|\xi^h\right|_2\le\left(1+N\,\mathrm{Re}^2\,|f|_*\right)\left|\eta^h\right|_2+\left(2N\,\mathrm{Re}^2\,|f|_*c\right)\sqrt{|\ln h|}\left|\psi-\psi^H\right|_1.
$$

The triangle inequality ($|\psi-\psi^h|_2\le|\xi^h|_2+|\eta^h|_2$) implies

$$
\left|\psi-\psi^h\right|_2\le\left(2+N\,\mathrm{Re}^2\,|f|_*\right)\left|\eta^h\right|_2+\left(2N\,\mathrm{Re}^2\,|f|_*c\right)\sqrt{|\ln h|}\left|\psi-\psi^H\right|_1. \tag{26}
$$

Hence, we have the following estimates:

$$
\left|\psi-\psi^h\right|_2\le C_1\inf_{w^h\in X^h}\left|\psi-w^h\right|_2+C_2\sqrt{|\ln h|}\cdot\left|\psi-\psi^H\right|_1. \qquad \blacksquare
$$

COROLLARY 5.1. *Let $X^{h,H}$ be the Clough-Tocher elements. Then ψ^h satisfies*

$$
\left|\psi-\psi^h\right|_2\le C_1h^2+C_2\sqrt{|\ln h|}H^3.
$$

6. SUMMARY

Two-level method for the stream function formulation of the Navier-Stokes equations was discussed. The method is important because of the superlinear scaling between the coarse and fine grids.

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