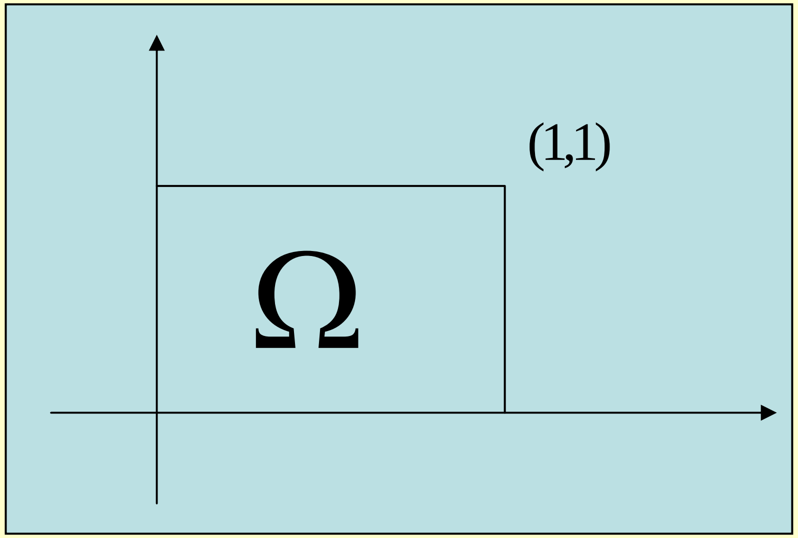
Galerkin's Method Example:

$$-\Delta u = f \quad in \quad \Omega$$
$$u = 0 \quad on \quad \Gamma$$



where $f(x,y) = 2(x^2 + y^2 - x - y)$

Use Galerkin's method with

$$S_0 = Span\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$$
$$\varphi_i(x,y) = \sin(i\pi x)\sin(i\pi y)$$

Solution:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} l(\varphi_1) \\ l(\varphi_2) \\ l(\varphi_3) \end{bmatrix}$$

$$a(\varphi_i, \varphi_j) = \int_0^1\int_0^1 \{\varphi_{i,x}\varphi_{j,x} + \varphi_{i,y}\varphi_{j,y}\}dxdy$$

$$a(\varphi_1,\varphi_1) = \int_0^1\int_0^1 \left(\{\pi\cos(\pi x)\sin(\pi x)\}^2 + \{\pi\cos(\pi y)\sin(\pi x)\}^2\right)dxdy = \frac{\pi^2}{2}$$

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$$a(\varphi_3,\varphi_3) = \int_0^1\int_0^1 \left(\{3\pi\cos(3\pi x)\sin(3\pi x)\}^2 + \{3\pi\cos(3\pi y)\sin(3\pi x)\}^2\right)dxdy = \frac{9\pi^2}{2}$$

$$l(\varphi_i) = \int_0^1\int_0^1 \varphi_i f dxdy$$

$$l(\varphi_1) = \int_0^1\int_0^1 \sin(\pi x)\sin(\pi y) \bullet 2(x^2 + y^2 - x - y)dxdy = -\frac{32}{\pi^4}$$
$$l(\varphi_2) = 0$$
$$l(\varphi_3) = -\frac{32}{81\pi^4}$$

$$\begin{bmatrix} \frac{\pi^2}{2} & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ 0 & 0 & \frac{9\pi^2}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{32}{\pi^4} \\ 0 \\ -\frac{32}{81\pi^4} \end{bmatrix}$$

$$c_1 = -\frac{64}{\pi^6}, c_2 = 0, c_3 = -\frac{64}{729\pi^6}$$

Exact Solution: $u(x, y) = -(x^2 - x)(y^2 - y)$

Approximation: $u^*(x, y) = -\frac{64}{\pi^6}\left(\sin(\pi x)\sin(\pi y) + \frac{1}{729}\sin(3\pi x)\sin(3\pi y)\right)$

$u(0.5, 0.5) = -0.0625$ $\qquad$ $u^*(0.5, 0.5) = -0.0666617$