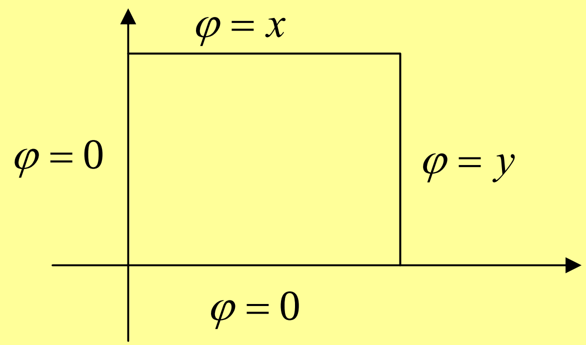
Chapter 1: Finite Difference Method for Poisson Equation

Example:

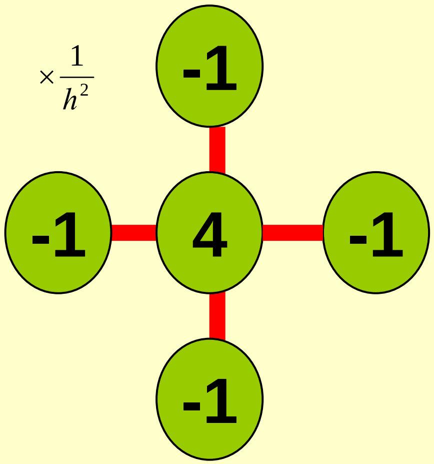
$$-\Delta u = 0 \quad \text{in} \quad \Omega$$

$$u = \varphi \quad \text{on} \quad \partial\Omega$$



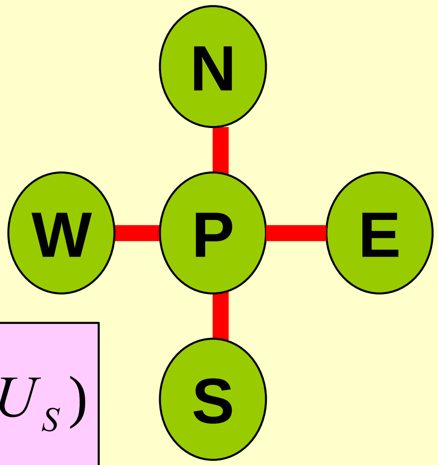
$$-\Delta_h U = -\frac{U_{i+1,j} + U_{i-1,j} - 4U_{i,j} + U_{i,j+1} + U_{i,j-1}}{h^2} = 0$$

Discrete Laplace Operator



5 point-scheme

5 point stencil



$$\frac{1}{h^2}(-U_E - U_W + 4U_P - U_N - U_S)$$

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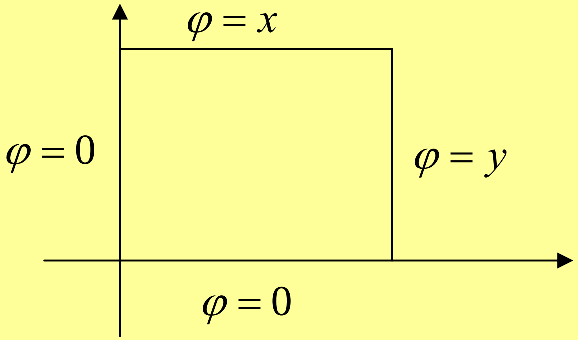


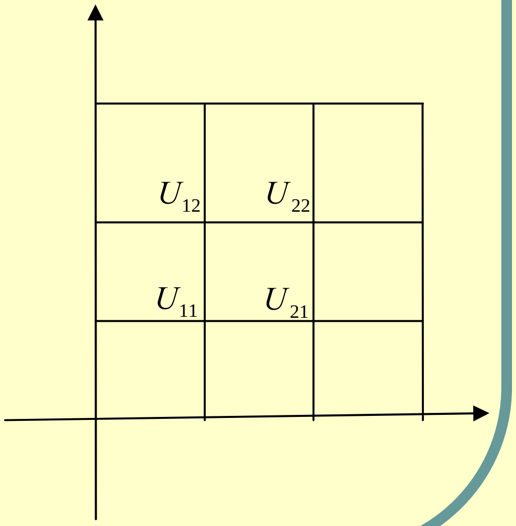
$$-\Delta_h U = -\frac{U_{i+1,j} + U_{i-1,j} - 4U_{i,j} + U_{i,j+1} + U_{i,j-1}}{h^2} = 0$$

$$i=1, j=1: \quad -U_{12} - U_{10} + 4U_{11} - U_{21} - U_{01} = 0$$

$$i=1, j=2: \quad -U_{13} - U_{11} + 4U_{12} - U_{22} - U_{02} = 0$$

$$i=2, j=1: \quad -U_{22} - U_{20} + 4U_{21} - U_{31} - U_{11} = 0$$

$$i=2, j=2: \quad -U_{23} - U_{21} + 4U_{22} - U_{32} - U_{12} = 0$$



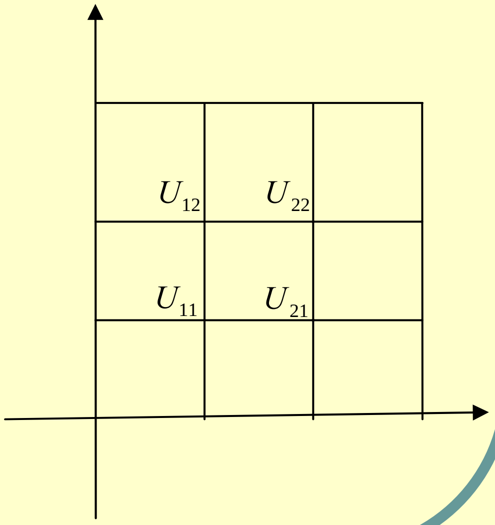
Chapter 1: Finite Difference Method for Poisson Equation

$$i=1, j=1: \quad -U_{12}-U_{10}+4U_{11}-U_{21}-U_{01}=0$$

$$i=1, j=2: \quad -U_{13}-U_{11}+4U_{12}-U_{22}-U_{02}=0$$

$$i=2, j=1: \quad -U_{22}-U_{20}+4U_{21}-U_{31}-U_{11}=0$$

$$i=2, j=2: \quad -U_{23}-U_{21}+4U_{22}-U_{32}-U_{12}=0$$



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$$i=1, j=1: \quad -U_{12} - U_{10} + 4U_{11} - U_{21} - U_{01} = 0$$

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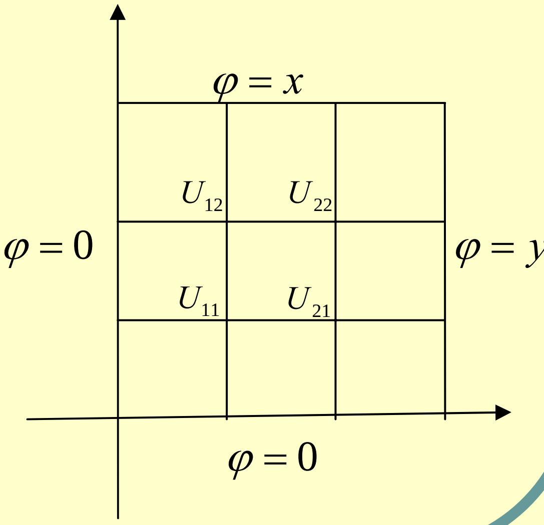

Chapter 1: Finite Difference Method for Poisson Equation

$$i=1, j=1: \quad -U_{12} - \cancelto{0}{U_{10}} + 4U_{11} - U_{21} - \cancelto{0}{U_{01}} = 0$$

$$i=1, j=2: \quad -U_{13} - U_{11} + 4U_{12} - U_{22} - \cancelto{0}{U_{02}} = 0$$

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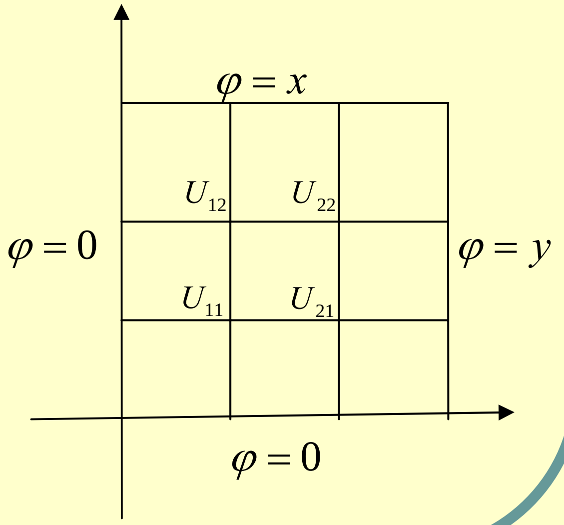


Chapter 1: Finite Difference Method for Poisson Equation

$$i=1, j=1: \quad -U_{12} - U_{10} + 4U_{11} - U_{21} - U_{01} = 0$$

$1/3$

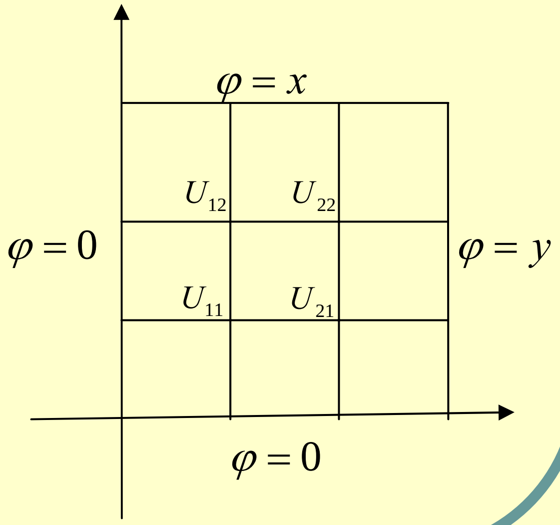
$$i=1, j=2: \quad -U_{13} - U_{11} + 4U_{12} - U_{22} - U_{02} = 0$$

$1/3$

$$i=2, j=1: \quad -U_{22} - U_{20} + 4U_{21} - U_{31} - U_{11} = 0$$

$$i=2, j=2: \quad -U_{23} - U_{21} + 4U_{22} - U_{32} - U_{12} = 0$$

$2/3$ $2/3$



Chapter 1: Finite Difference Method for Poisson Equation

$$i=1, j=1: \quad -U_{12} - \cancelto{0}{U_{10}} + 4U_{11} - U_{21} - \cancelto{0}{U_{01}} = 0$$

$$i=1, j=2: \quad -\cancelto{1/3}{U_{13}} - U_{11} + 4U_{12} - U_{22} - \cancelto{0}{U_{02}} = 0$$

$$i=2, j=1: \quad -U_{22} - \cancelto{0}{U_{20}} + 4U_{21} - \cancelto{1/3}{U_{31}} - U_{11} = 0$$

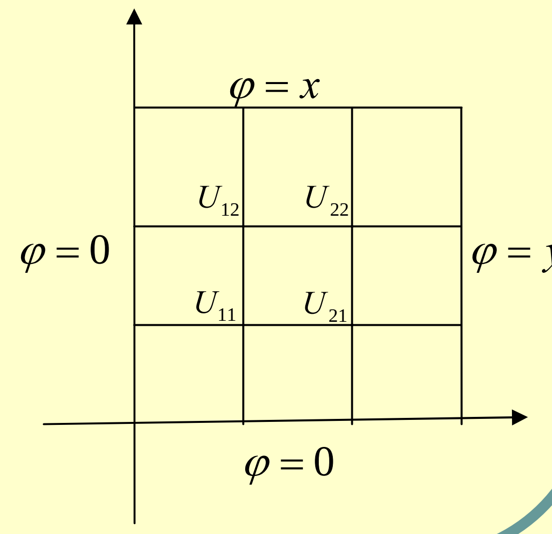
$$i=2, j=2: \quad -\cancelto{2/3}{U_{23}} - U_{21} + 4U_{22} - \cancelto{2/3}{U_{32}} - U_{12} = 0$$

$$-U_{12} - \quad + 4U_{11} - U_{21} - \quad = 0$$

$$-U_{11} + 4U_{12} - U_{22} - \quad = 1/3$$

$$-U_{22} \quad + 4U_{21} \quad - U_{11} = 1/3$$

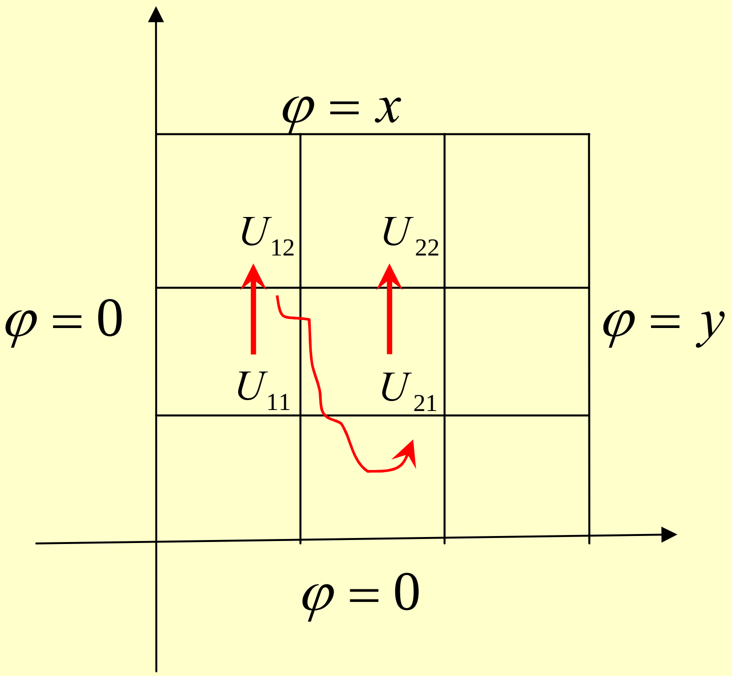
$$-U_{21} + 4U_{22} \quad - U_{12} = 4/3$$



Numbering

$$
\begin{aligned}
&-U_{12} - \quad + 4U_{11} - U_{21} - \quad = 0 \\
&-U_{11} + 4U_{12} - U_{22} - \quad = 1/3 \\
&-U_{22} \quad + 4U_{21} \quad - U_{11} = 1/3 \\
&-U_{21} + 4U_{22} \quad - U_{12} = 4/3
\end{aligned}
$$

$$
\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}
\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 4/3 \end{bmatrix}
$$



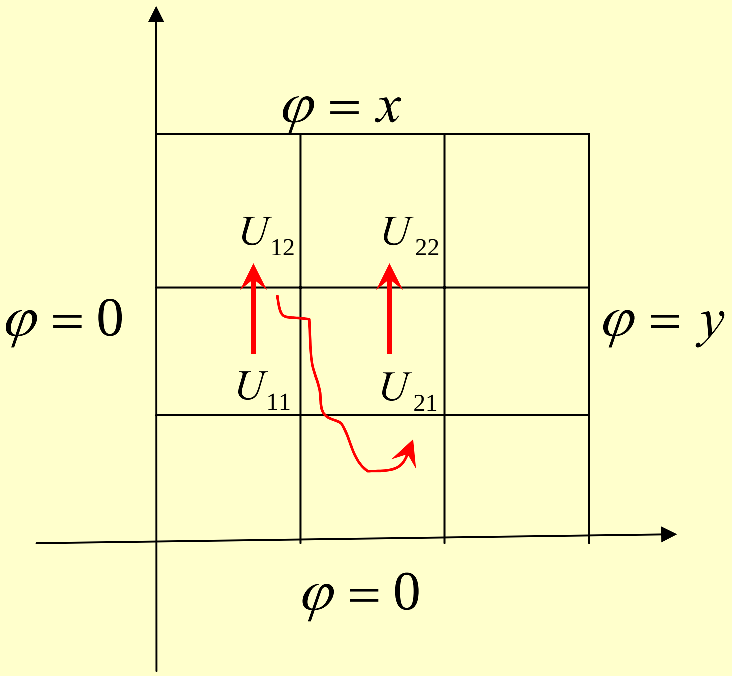
Numbering

$$-U_{12} - \quad +4U_{11} - U_{21} - \quad = 0$$

$$-U_{11} + 4U_{12} - U_{22} - \quad = 1/3$$

$$-U_{22} \quad +4U_{21} \quad -U_{11} = 1/3$$

$$-U_{21} + 4U_{22} \quad -U_{12} = 4/3$$

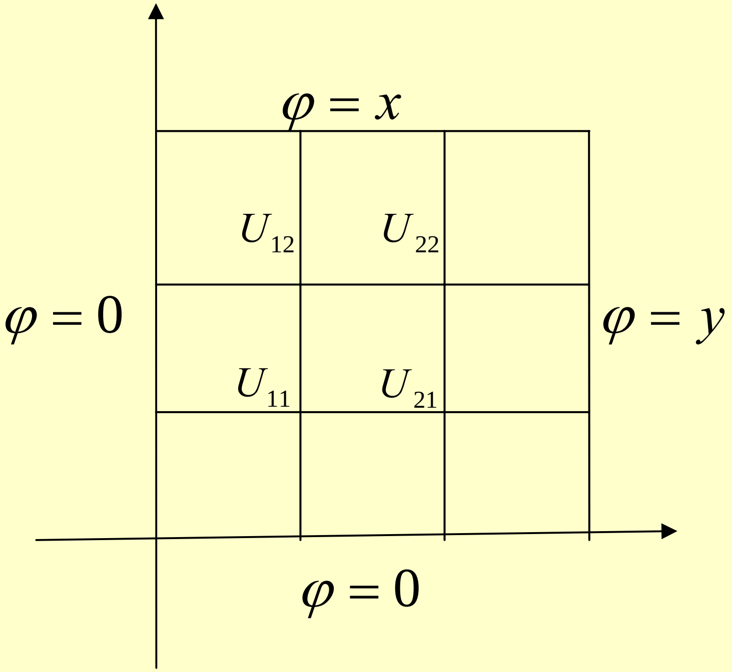


$$\left[\begin{array}{cc:cc} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ \hdashline -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{array}\right]\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 4/3 \end{bmatrix}$$

$$AU = b$$

Solving the Linear System

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 4/3 \end{bmatrix}$$



$$A^{-1} = \frac{1}{24} \begin{bmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1/3 \\ 1/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \\ 4/9 \end{bmatrix}$$

Exact Solution

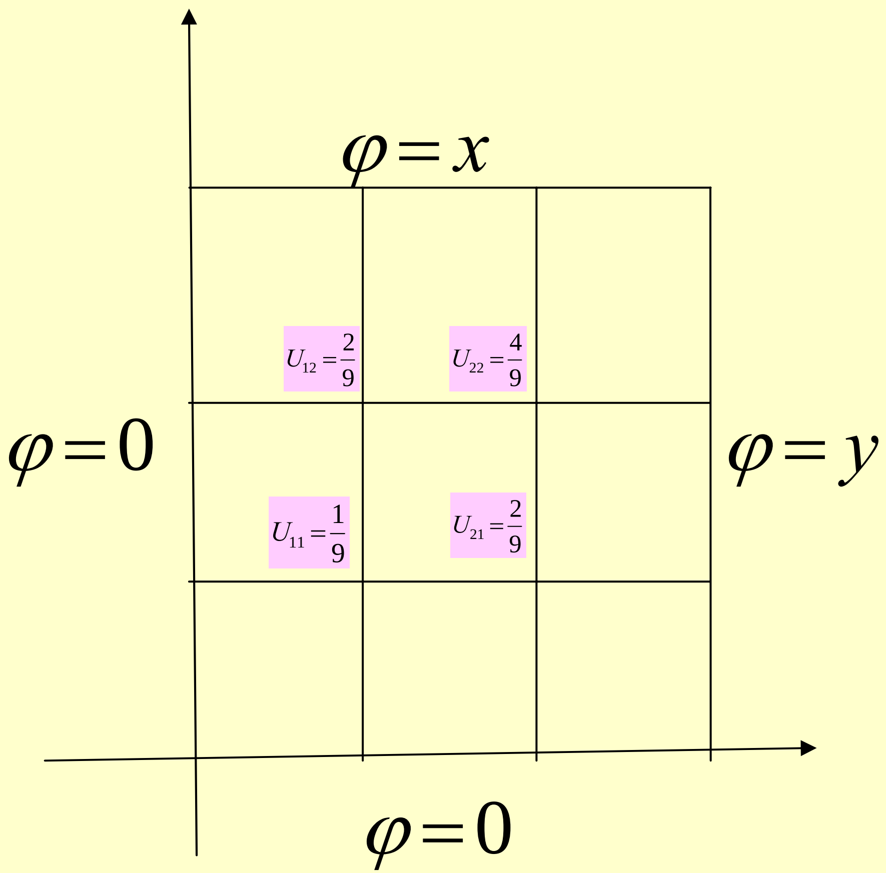
Example :

$$-\Delta u = 0 \quad \text{in} \quad \Omega$$

$$u = \varphi \quad \text{on} \quad \partial\Omega$$

Exact Solution

$$u(x, y) = xy$$



FDM →

$$\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} 1/9 \\ 2/9 \\ 2/9 \\ 4/9 \end{bmatrix}$$