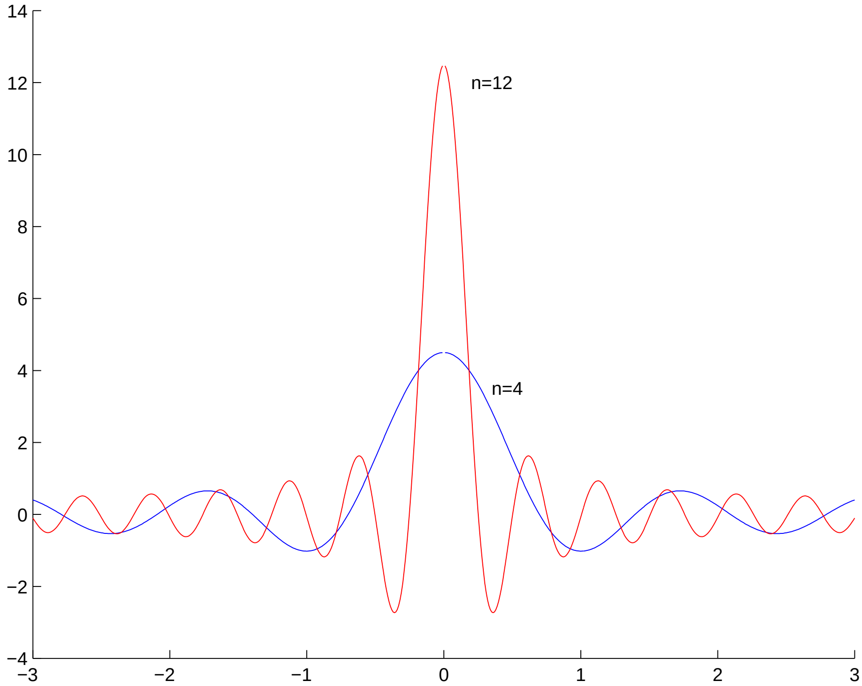
Partial Differential Equations of Applied Mathematics

Lecture Notes, Math 713

Fall, 2003



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Preface

We have comprehended some auspicious characters.

Sheriff Dogberry, in *Much Ado about Nothing*

These lecture notes have arisen over a period of years teaching courses in partial differential equations at the University of Minnesota and Yale University. They are, by and large, a plain vanilla approach to the subject, emphasizing emphasize basic elements of partial differential equations and applied analysis and their applications to the basic equations of mathematical physics and applied mathematics. They are intended to be selective rather than encyclopedic, and to illustrate the interrelationship between mathematics and physics. The main prerequisites for the course are a solid foundation in undergraduate analysis, vector analysis, some background in physics, and an interest in the application of mathematics to natural phenomena. They are aimed at students in mathematics, physics, and engineering.

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Chapter 1

The Heat Equation

1.1 The flow of heat

The heat equation describes the flow of heat in a homogeneous isotropic medium. In three space dimensions it is

$$\frac{\partial u}{\partial t} = K\Delta u, \qquad \Delta u = u_{xx} + u_{yy} + u_{zz}. \tag{1.1}$$

The operator Δ is called the *Laplacian* and plays a fundamental role in mathematical physics and in partial differential equations. The constant K is determined by physical properties of the medium, and is called the diffusion coefficient.

Throughout these notes we denote the time variable by t, and the spatial variable by x. When $x \in \mathbb{R}^3$, we write $x = (x_1, x_2, x_3)$, and we abbreviate the partial derivatives of u by

$$u_t = \frac{\partial u}{\partial t}, \qquad u_j = \frac{\partial u}{\partial x_j},$$

etc.

The heat equation models not only the flow of heat energy through a medium, but also models diffusion processes, such as *Brownian motion*, the statistical motion of very small particles suspended in a fluid. We first derive the heat equation from the point of view of continuum mechanics, then discuss methods of solution and some of its properties. Later we discuss the relationship of the heat equation to Brownian motion and diffusion processes in statistical physics.

Let $u(x, t)$ denote the temperature in a medium at the point x at time t, and let Q denote the density of heat energy. The heat energy is a function of the temperature: $Q = Q(x, u)$ per unit volume. We should expect that $Q_u > 0$, and the simplest assumption that we can make is that Q is linearly proportional to u with a fixed, positive, spatially independent constant c: $Q = cu$. The constant $c > 0$ is called the *specific heat*. In reality the specific heat may itself depend on x and u, but this will lead to more complicated equations; so for now we limit the discussion to the simplest case.

If C denotes a domain (an open set) in $\mathbb{R}^3$, then the total heat energy in C is given by the volume integral

$$\iiint_C cu\,dx,$$

and the rate of change of heat energy inside C is

$$\frac{\partial}{\partial t}\iiint_C cu\,dx.$$

If the temperature is not uniform the heat will flow from hotter to colder regions. This is expressed mathematically by saying that the heat flux across an oriented surface is opposite to the gradient of the temperature on that surface. Denote the heat flux by a vector field $\vec{F}$, and think of heat as a kind of invisible fluid and $\vec{F}$ as the flux of that fluid per unit area in the direction of $\vec{F}$. Thus, the amount of heat per unit time flowing across a surface S in space is given by the integral

$$\iint_S \vec{F}\cdot d\vec{S}$$

Here, $d\vec{S}$ is the oriented element of surface area $d\vec{S} = \widehat{n}\,dS$, where $\widehat{n}(x)$ is the normal unit vector to the surface S and dS is the infinitesimal area element.

Since heat energy moves from warmer to cooler regions, the flux is in the opposite direction of the temperature gradient and can be written

$$F = -k(u)\nabla u, \qquad \nabla u = (u_1, u_2, u_3)$$

where $k(u) > 0$. Let us assume our material is a simple one, with $k(u)$ equal to a positive constant, k.

By conservation of heat energy, the rate of *increase* in heat energy inside C is equal to the flux of energy flowing *in* across the boundary of C, (denoted by ∂C). Thus

$$\frac{\partial}{\partial t} \iiint_C cu\,dx = -\iint_{\partial C} F \cdot d\vec{S} = \iint_{\partial C} k\nabla u \cdot\, d\vec{S}.$$

By the divergence theorem,

$$\iint_S k\nabla u \cdot d\vec{S} = k \iiint_C \Delta u\,dx,$$

where $\Delta u = \operatorname{div} \nabla\, u$. Combining these two results we find

$$\iiint_C cu_t - k\Delta u\,dx = 0 \tag{1.2}$$

for any smoothly bounded domain C.

It is important to emphasize that this conservation law holds for *any* domain C with a smooth boundary contained in the domain of definition of u. For simplicity, we assume that the integrand in (1.2) is continuous. If it is not identically zero, then it is, say, positive at some point x_0. By continuity it is positive in some small ball B containing x_0. Applying the result (1.2) to the domain B we obtain a contradiction, for the integrand is everywhere positive, yet the integral vanishes. Thus the integrand cannot be positive anywhere; and by a similar argument, it cannot be negative anywhere either, so it must be identically zero. Hence we obtain the heat equation (1.1) with the diffusion constant $K = k/c$.

1.2 The fundamental solution

We now turn to the problem of constructing a solution of the initial value problem for the heat equation in the case of one space variable with $K = 1$:

$$u_t = u_{xx}, \qquad u(x, 0) = f(x). \tag{1.3}$$

Recall that the solution to a linear system of ordinary differential equations $\dot{x} = Ax,\ x(0) = x_0$ is given by $x(t) = e^{tA}x_0$, where the exponential of a matrix is defined by a power series expansion. The corresponding formal solution to the heat equation would be $u(t) = e^{t\Delta}u_0$ where u_0 and $u(t)$ belong to some class of functions defined on $\mathbb{R}^n$. The question we want to resolve is how to interpret the formal expression $e^{t\Delta}$ where Δ denotes the Laplacian. The representation of this operator leads to the construction of what is known as the *fundamental solution* of the heat equation.

We shall show that the solution of the heat equation in $\mathbb{R}^1$ can be written in the form

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy.$$

Such an integral is called a *convolution.* If we denote $G_t(x) = G(x,t)$, then we may formally write

$$e^{t\Delta}f = G_t * f.$$

It is immediate that the function $G(x,t)$ must be a solution of the heat equation for $t > 0$ and have the property that

$$\lim_{t\to 0+}\int_{-\infty}^{\infty} G(x-y,t)f(y)\,dy = f(x), \tag{1.4}$$

at least for some large class of functions f. Moreover, since the solution of the initial value problem with $f(x) = 1$ is $u(x,t) = 1$, we should also expect that

$$\int_{-\infty}^{\infty} G(x-y,t)\,dy = 1$$

for all $x,\ t > 0$. The function $G(x,t)$ is the fundamental solution of the heat equation in one dimension.

One common technique for finding special solutions of partial differential equations is to reduce the equation to an ordinary differential equation can be solved explicitly. We do that in the present case by using a device that works in many cases, linear as well as nonlinear. Suppose in the heat equation we rescale the variables by letting $x \to \lambda x$ and $t \to \lambda^2 t$. More explicitly, we consider the transformations of u defined by

$$T_\lambda u(x,t) = u(\lambda x, \lambda^2 t) \tag{1.5}$$

The transformations $T_\lambda,\ \lambda > 0$ form a group of transformations, called the *dilation group*. It is a simple exercise to show that T_λ maps solutions of the heat equation to solutions.

Let us see how the solution of the initial value problem transforms under such a rescaling. The solution of (1.3) with initial data $T_\lambda f$ is $T_\lambda u$. Hence

$$\int_{-\infty}^{\infty} G(\lambda x - y, \lambda^2 t) f(y)\, dy = \int_{-\infty}^{\infty} G(\lambda(x-z), \lambda^2 t) f(\lambda z)\lambda\, dz$$

$$= \int_{-\infty}^{\infty} \lambda G(x-z, t) f(\lambda z)\, dz,$$

and

$$\lambda G(\lambda x, \lambda^2 t) = G(x,t). \tag{1.6}$$

Definition A function $u(x,t),\ x \in \mathbb{R}^n,\ t > 0$ is homogeneous of degree α with respect to the action of the dilation group if $T_\lambda u = \lambda^\alpha u$ for all $\lambda > 0$: $u(\lambda x, \lambda^2 t) = \lambda^\alpha u(x,t)$

The argument above shows that the fundamental solution is homogeneous of degree -1. The following result may be proved by the same general argument.

Theorem 1.2.1 *The fundamental solution of the heat equation in n space dimensions is homogeneous of degree $-n$.*

Now (1.6) holds for all $\lambda > 0$; in particular, for any fixed value of t we may choose $\lambda = t^{-1/2}$. We then see that G has the property that

$$G(x,t) = \frac{1}{\sqrt{t}}\varphi(\xi),$$

where

$$\xi = \frac{x}{\sqrt{t}}, \qquad \varphi(\xi) = u(\xi, 1).$$

The variable ξ is called a *similarity variable.*

We also leave it as an exercise to show that φ satisfies the ordinary differential equation

$$\varphi'' + \frac{1}{2}(\xi\varphi)' = 0, \tag{1.7}$$

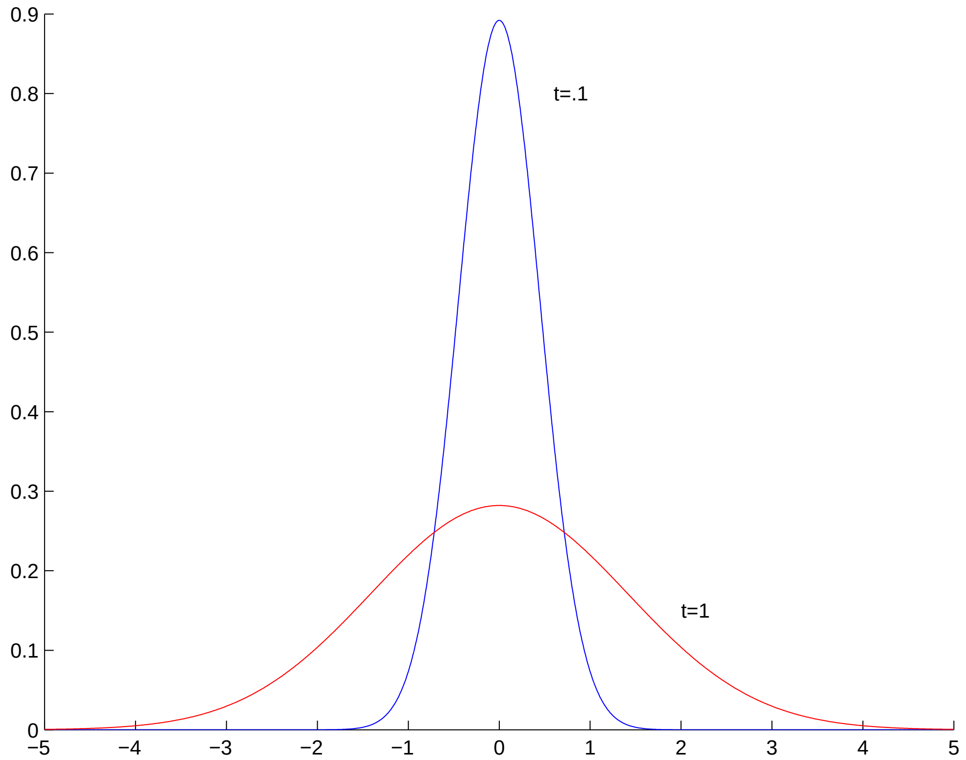


Figure 1.1: The fundamental solution of the heat equation evaluated at $t = .1$ and $t = 1$.

and that a solution of this equation is

$$\varphi = Ae^{-\frac{1}{4}\xi^2}, \tag{1.8}$$

where A is a constant of integration. The corresponding solution of the heat equation is

$$\frac{A}{\sqrt{t}} e^{-x^2/4t}. \tag{1.9}$$

It is a straightforward matter to check that this expression satisfies the heat equation.

The integral of this function over the real line is constant in t, and A may be chosen so that the integral is precisely 1. We then obtain

$$G(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \tag{1.10}$$

as the fundamental solution of the heat equation on the line.

Theorem 1.2.2 *Let $f(x)$ be bounded and continuous on the real line. The solution of the initial value problem* (1.3) *is given by*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\,dy. \tag{1.11}$$

Proof: We leave it as an exercise to show that (1.11) is a solution of the heat equation, and show here that

$$\lim_{t\downarrow 0} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\,dy = f(x)$$

at every point of continuity x of f. Since the fundamental solution is normalized so that its integral over the real line is 1, we have

$$\begin{aligned}\Delta(x,t) &= \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} f(y)\,dy - f(x)\\ &= \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} (f(y) - f(x))\,dy.\end{aligned}$$

We must show that $\Delta(x,t) \to 0$ as $t \downarrow 0$. For fixed x and any $\varepsilon > 0$, we choose $\delta > 0$ so that

$$|f(y) - f(x)| < \varepsilon \quad \text{for all} \quad |x - y| < \delta.$$

Then

$$\begin{aligned}\Delta(x,t) = \int_{|x-y|<\delta} &\frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} (f(y) - f(x))\,dy\\ &+ \int_{|x-y|\ge\delta} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} (f(y) - f(x))\,dy\\ =&\Delta_1(x,t) + \Delta_2(x,t).\end{aligned}$$

Now

$$|\Delta_1(x,t)| \le \int_{|x-y|<\delta} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} |f(y) - f(x)|\,dy < \varepsilon.$$

Moreover, since f is bounded, say $|f(y)| \leq M$ for all y,

$$\begin{aligned}
\Delta_2(x,t) \leq & 2M \int_{|x-y|\geq\delta} \frac{e^{-(x-y)^2/4t}}{\sqrt{4\pi t}} dy \\
= & 4M \int_{\delta}^{\infty} \frac{e^{-y^2/4t}}{\sqrt{4\pi t}} dy = \frac{4M}{\sqrt{4\pi}} \int_{\delta/4\sqrt{t}}^{\infty} e^{-y^2} dy \\
\to & 0 \text{ as } t \downarrow 0.
\end{aligned}$$

The above argument shows that $\limsup_{t\to 0+} \Delta(x,t) \leq \varepsilon$ for any $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, $\Delta(x,t)$ must tend to zero as $t \downarrow 0$. ■.

The fundamental solution of the heat equation in $\mathbb{R}^n$ is

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}, \qquad r^2 = x_1^2 + \ldots x_n^2. \tag{1.12}$$

The proof is left as an exercise.

We now turn to a discussion of some of the properties of the solution of the heat equation. In an appropriate sense, these properties are characteristic of a more general class partial differential equations known as second order parabolic equations. The reason for the terminology will become apparent later.

Since the fundamental solution of the heat equation is strictly positive and has integral 1 for all $t > 0$, we immediately have the following result, known as the *Strong Maximum Principle*:

Theorem 1.2.3 *Let $a \leq f(x) \leq b$ for all real x, and suppose that f is not identically constant. Then the solution* (1.11) *of the heat equation satisfies the strict inequalities $a < u(x,t) < b$ for all $t > 0$.*

Proof: Since the integral of the fundamental solution is identically 1,

$$u(x,t) - a = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} (f(y) - a)\, dy.$$

The integrand is non-negative and strictly positive on any interval on which $f > a$; hence it can never be zero for any x or any $t > 0$. Hence $u(x,t) > a$ for all $t > 0$. Similarly $u(x,t) < b$ for all $t > 0$. ■

The strong maximum principle for the heat equation is a precise mathematical formulation of the physical observation that heat flows from warmer to cooler regions. In the absence of external heat sources, the temperature cannot rise above or below the original ambient values. In this sense the heat equation accurately reflects the physics of heat. In another sense, however, it is an unphysical model; for, by the same argument we used above, we can prove the following:

Proposition 1.2.4 *Under the evolution of the heat equation, disturbances propagate infinitely fast.*

The *support* of a function f, denoted by $\text{supp}\,(f)$, is the set of points x for which $f(x) \neq 0$. Suppose $f(x) \geq 0$ everywhere on $\mathbb{R}$ and $\text{supp}(f) \subset K$, where K is a compact subset of the real line. Let u satisfy (1.3). Then by the same argument we used to prove the strong maximum principle,we can show that u is strictly positive for all x and all $t > 0$. In other words, heat propagates infinitely fast.

This is clearly a nonphysical phenomenon. Thus, the heat equation is not really a fundamental equation of physics; rather, it is an approximate model, albeit a very good one. We will return to this later in §1.4.

The strong maximum principle holds for more general parabolic equations in n spatial dimensions. For example, consider the partial differential operator

$$L[u] = \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial t} + c(x,t)u.$$

L is said to be *uniformly parabolic* in a domain $\Omega \subset \mathbb{R}^{n+1}$ if the coefficients a_{jk} satisfy the inequality

$$\sum_{j,k=1}^{n} a_{jk}(x,t)\xi_j\xi_k \geq \mu \sum_{j=1}^{n} \xi_j^2$$

for all $(x,t) \in \Omega$ for some $\mu > 0$.

Let Ω_T be the domain

$$\Omega_T = \{(x,t) \,:\, x \in \Omega,\ 0 \leq t \leq T\}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$. Then

Theorem 1.2.5 *Let* $c(x,t) \leq 0$ *in* Ω_T *and suppose that* u *satisfies the uniformly parabolic partial differential inequality*

$$L[u] \geq 0 \qquad (x,t) \in \Omega_T.$$

Suppose that $u \leq M$ *in* Ω_T *and* $u(x_0, T) = M$ *at an interior point* $x_0 \in \Omega$. *Then* $u \equiv M$ *for all* $x \in \Omega$ *and all* $0 \leq t \leq t_0$.

The maximum principle also holds for nonlinear parabolic equations as well, under appropriate conditions. For example, the coefficients a_{jk}, b_j and c in the definition of the operator L can all depend on the solution u and its derivatives u_{x_j} and $u_{x_j x_k}$. A very readable account of maximum principles for partial differential equations is given in the book by Protter and Weinberger [15].

The heat equation does a lot of *smoothing*. Even if the initial data f is only bounded and measurable, the solution of the heat equation u is real analytic in x and t for $t > 0$. By that we mean that the solution can be expanded in a convergent Taylor series about any point x_0, t_0 where $t_0 > 0$. We will prove a somewhat weaker result here:

Theorem 1.2.6 *Let* f *be a bounded measurable function on the real line. Then the solution* (1.11) *is infinitely differentiable in both* x *and* t *for all* $t > 0$.

Proof: The fundamental solution is infinitely differentiable in both variables, and it, together with all its derivatives in x and t, decay exponentially fast as $x \to \pm\infty$ for any $t > 0$. Therefore we may interchange the process of differentiation and integration in the integral (1.11). The same argument holds for the solution in $\mathbb{R}^n$. ■

In Figure 1.2 we see the graph of the solution of the initial value problem for the heat equation with discontinuous initial data:

$$H(x) = \begin{cases} 0 & -\infty < x < 0; \\ 1 & 0 < x < \infty. \end{cases} \tag{1.13}$$

The function H is known as the *Heaviside step function.*

Analyticity of the solution of the heat equation actually follows from the analyticity of the fundamental solution.

We consider now the solution of the inhomogeneous initial value problem

$$u_t = u_{xx} + h, \qquad u(x,0) = f$$

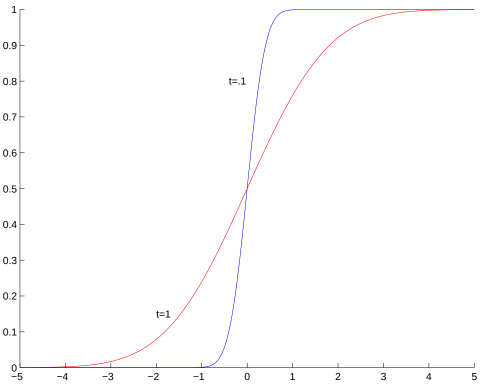


Figure 1.2: Solution of the heat equation on the line with initial $u(x, 0) = H(x)$ at times $t = .1$, 1. The solution is continuous and lies strictly between 0 and 1 for all positive time.

where f is defined on the real line, and $h = h(x, t)$ is defined on $t > 0$. There is a standard method of solving inhomogeneous linear equations, known as *Duhamel's principle*. It goes like this.

Consider an abstract equation of the form $u_t = Au+h$, $u(0) = f$. It is useful to regard the heat equation as an infinite dimensional system of ordinary differential equations, with $A = d^2/dx^2$. If A were a matrix on a finite dimensional vector space, the solution would be given by

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A} f(s)\, ds. \tag{1.14}$$

In the present case, $e^{tA}f$ is precisely the integral operator given by (1.11). For the heat equation, (1.14) should be interpreted as

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\, dy$$
$$+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-(x-y)^2/4(t-s)} h(y, s)\, dy\, ds$$

Having derived this expression formally, one can check directly that it gives the required solution.

1.3 Fourier series & separation of variables.

We noted above that partial differential equations are infinite dimensional in character, and that, in particular, the heat equation has an infinite number of solutions. We apply that observation now to derive the solution of the initial value problem for the heat equation on a finite interval. This problem was originally posed and solved by J. J. FOURIER, *Theorie Analytique de la Chaleur*, and led to the development of Fourier series and integrals, one of the most significant tools of pure and applied mathematics.

Suppose we want to solve the initial value problem for the heat equation on a finite interval. For convenience, we take scale the spatial variable so that the interval is $[0, \pi]$. Thus we have:

$$u_t = u_{xx}, \quad 0 \le x \le \pi; \qquad u(x, 0) = f(x) \tag{1.15}$$

together with the boundary conditions

$$u(0, t) = u(\pi, t) = 0.$$

Many other situations are possible; but we shall treat only this simple case. For a more extensive discussion of the method of separation of variables, see PINSKY [14], SOMMERFELD [23], WEINBERGER [25]. As in the problem on the infinite interval, we begin by constructing special solutions of the homogeneous problem. We look for solutions of $u_t = u_{xx}$ of the form

$$u(x, t) = X(x)T(t).$$

This is the so-called process of *separation of variables*.

Substituting this expression into the heat equation we obtain the relation $T'X = X''T$, which we write as

$$\frac{T'}{T} = \frac{X''}{X}.$$

The only way such a relationship can hold for arbitrary x and t is for both ratios to be constant. For reasons that will quickly become clear, we take this constant to be $-\gamma^2$. Then we have

$$T' = -\gamma^2 T, \qquad X'' + \gamma^2 X = 0.$$

The general solutions of this pair of equations are

$$T(t) = T_0 e^{-\gamma^2 t}, \qquad X(x) = A \sin \gamma x + B \cos \gamma x.$$

In particular, T is a bounded, in fact, exponentially decreasing, function of time. The constant γ is not arbitrary, but is restricted by the requirement that the solutions satisfy the boundary condition $X(0) = X(\pi) = 0$. The boundary condition $X(0) = 0$ implies that $B = 0$, while the boundary condition $X(\pi) = 0$ implies that $\sin \gamma\pi = 0$, hence that $\gamma = n$, where n is an integer. Since $\sin -\gamma\pi = -\sin \gamma\pi$, there is no loss in generality if we restrict n to be positive.

Hence we have constructed an *infinite* number of independent solutions of the heat equation, namely

$$u_n(x,t) = e^{-n^2 t} \sin\, nx.$$

Since the equation is linear and homogeneous, we may form more general solutions of the heat equation by taking a linear superposition of such solutions:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx.$$

Since the exponential terms converge very rapidly to zero for $t > 0$, there is no problem proving this series converges and satisfies the heat equation under very minimal restrictions on the coefficients a_n. It also satisfies the boundary conditions.

The only remaining question is whether the a_n can be chosen so as to satisfy the initial conditions, that is, whether the initial data f can be represented as an infinite sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

This question has to do with the *completeness* of the family of functions $\{\sin nx\}$ on the interval $[0,\pi]$. Since a sine series automatically represents an odd, 2π periodic function, we consider the odd, 2π-periodic extension of f. The coefficients a_n are given by (cf. exercises)

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\, nx\, dx. \tag{1.16}$$

Since the method of Fourier series has been extensively discussed elsewhere, we will not go into a great amount of detail on the method. However, let us prove here that the Fourier sine series converges to f, given some conditions on the regularity of f.

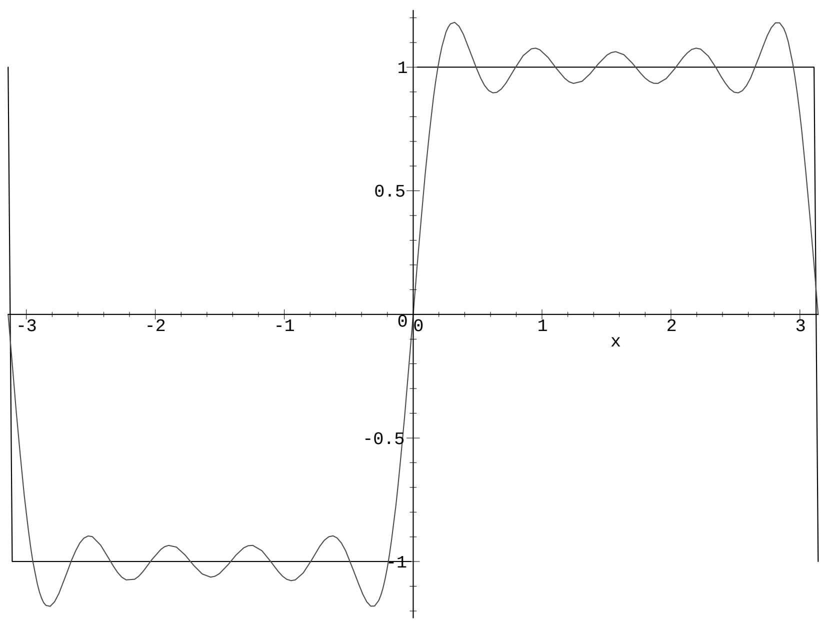


Figure 1.3: Partial sums S_1, S_2, and S_5 of the Fourier series for the step function (1.18). The overshooting is known as *Gibbs phenomenon.*

The partial sums of the Fourier sine series of f are

$$
\begin{aligned}
S_{n,f}(x) =& \frac{1}{\pi}\sum_{k=1}^{n}\sin kx \int_{-\pi}^{\pi} f(t)\sin\, kt\, dt \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sum_{k=1}^{n}\sin kx \sin kt\, dt. \qquad (1.17)
\end{aligned}
$$

The convergence of the Fourier series to the function is an issue of some complexity. For differentiable functions, the Fourier series converge uniformly to the function itself. But when the function is discontinuous, e.g.

$$
f(x) = \begin{cases} -1 & -\pi \le x \le 0; \\ 1 & 0 \le x \le \pi, \end{cases} \qquad (1.18)
$$

the partial sums of the Fourier series "overshoot" the function as shown in Figure (1.3). The overshooting of the Fourier partial sums contrasts with the behavior of the solution of the heat equation (Figure (1.2)), which strictly interpolates the discontinuity, positivity of the fundamental solution of the heat equation.

Now

$$
\sin kx \sin kt = \frac{\cos k(x-t) - \cos k(x+t)}{2}.
$$

We leave it as an exercise to show that

$$
D_n(\theta) := \frac{1}{2} + \sum_{k=1}^{n}\cos k\theta = \frac{1}{2}\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \qquad (1.19)
$$

The function $D_n(\theta)$ is called *Dirichlet's kernel*.

The partial sum S_n can therefore be written

$$
S_{n,f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)[D_n(t-x) - D_n(x+t)]\, dt.
$$

By changing variables in the two terms and using the fact that f is odd and 2π-periodic, we can rewrite S_n as

$$
S_{n,f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)[f(x+t) + f(x-t)]\, dt.
$$

Since the integral of $D_n(t)$ over $[-\pi, \pi]$ is π, we have

$$S_{n,f}(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(t) f_x(t)\, dt,$$

$$f_x(t) = \frac{f(x+t) + f(x-t) - 2f(x)}{2}. \tag{1.20}$$

Now note that

$$D_n(t) = \frac{1}{2}\left(\cos nt + \sin nt \cot \frac{t}{2}\right);$$

hence

$$S_{n,f}(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_x(t) \cos nt + f_x(t) \cot \frac{t}{2} \sin nt\, dt.$$

The first integral on the right is the Fourier cosine coefficient of the function f_x, while the second term is the Fourier sine coefficient of

$$f_x(t) \cot \frac{t}{2}.$$

Using this observation, we prove:

Theorem 1.3.1 *Suppose that for fixed $x \in (-\pi, \pi)$ $f_x(t)/t$ is integrable on $(-\pi, \pi)$, i.e.*

$$\int_{-\pi}^{\pi} \left| \frac{f(x+t) + f(x-t) - 2f(t)}{t} \right| dt < +\infty.$$

Then $\lim_{n\to\infty} S_n(x) = f(x)$.

This criterion, due to Dini, holds if f is differentiable at x, but the criterion is far less restrictive.

Proof: Since $S_n - f$ is expressed as the Fourier coefficients of two integrable functions, the result follows immediately from the following result, known as the *Riemann-Lebesgue* lemma for Fourier coefficients:

Lemma 1.3.2 *The Fourier coefficients of a function $f \in L^1(-\pi, \pi)$ tend to zero with n.*

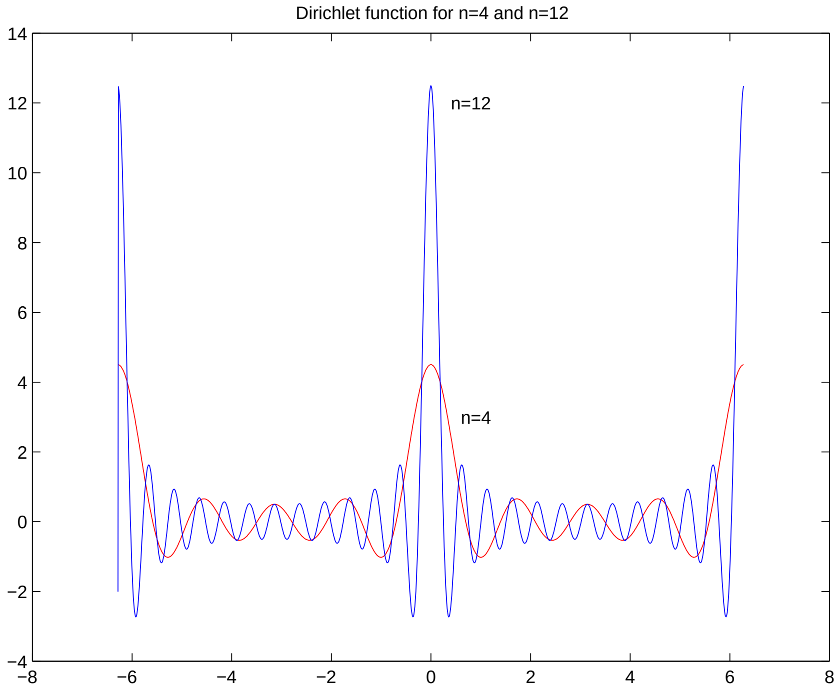


Figure 1.4: The Dirichlet kernel evaluated at $n = 4$, 12. The oscillation in the Dirichlet kernel gives rise to the Gibbs phenomenon.

The Riemann-Lebesgue lemma in turn follows from *Bessel's* inequality for orthonormal sequences, which we now prove. For functions $f,\ g \in L^2(-\pi, \pi)$ define

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx,$$

The vector space $L^2(-\pi, \pi)$ is a complex inner product space, with $(\ ,\ )$ as a dot or inner product. It is an infinite dimensional complex analog of the real Euclidean space $\mathbb{R}^3$. The analog of the ordinary Euclidean distance is given by

$$||f||^2 = (f, f).$$

The functions $\{e^{inx}\}$ form an *orthonormal sequence* on $L^2(-\pi, \pi)$ with respect to this inner product. That is,

$$(e^{inx},\ e^{imx}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \delta_{nm}$$

where δ_{nm} is the Kronecker delta.

Let $a_n = (f, e^{inx})$. If $||f|| < +\infty$, then a simple computation shows that

$$0 \le \left|\left| f - \sum_n a_n e^{inx} \right|\right|^2$$

$$= ||f||^2 - 2\sum_n |a_n|^2 + \sum_n |a_n|^2 = ||f||^2 - \sum_n |a_n|^2;$$

hence

$$\sum_n |a_n|^2 \le ||f||^2.$$

This is known as *Bessel's* inequality. An immediate consequence of this inequality is that the sum of the squares of the absolute values of the Fourier coefficients of an L^2 function is convergent, hence

$$\lim_{n\to\infty} a_n = 0.$$

We have given the proof here for the complex exponentials e^{inx}, but the proof for the sine functions is the same.

To extend the result to Fourier coefficients of L^1 functions, that is, to functions f satisfying the weaker condition

$$\int_{-\pi}^{\pi} |f|\, dx < +\infty,$$

we use a fact from Lebesgue integration theory that, given any function $f \in \mathcal{L}^1$ and given $\varepsilon > 0$, f can be decomposed into the sum of a bounded function and a function whose L^1 norm is less than ε. That is, $f = f_1 + f_2$, where,

$$||f_1||_\infty = \sup_x |f_1| < +\infty \quad \text{and} \quad \int |f_2|\, dx < \varepsilon.$$

The Fourier coefficients are correspondingly decomposed into $a_n = a_{n,1} + a_{n,2}$, corresponding to the Fourier coefficients of f_1 and f_2 respectively.

Since f_1 is bounded, it belongs to L^2, and its Fourier coefficients, $a_{n,1}$, tend to zero by the Bessel inequality. On the other hand,

$$|a_{n,2}| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f_2(x)e^{inx}\, dx\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_2(x)|\, dx < \varepsilon.$$

Hence

$$\limsup_{n\to\infty} |a_n| \leq \varepsilon$$

for any $\varepsilon > 0$, and the limit must in fact be zero. ■

Dini's test gives a simple criterion for the pointwise convergence of the partial sums of a Fourier series. For a more comprehensive discussion of the convergence of Fourier series, see A. Zygmund [28].

The history of Jean-Joseph Fourier (1768-1830) is a fascinating one. 'Citizen' Fourier served as Secretary of the Commission of Arts and Sciences in Napolean Bonaparte's expedition to Egypt in 1798. His tour of duty in Egypt affected his health, and when he returned to Grenoble, in the French Alps, "he covered himself with an excessive amount of clothing even in the heat of summer, and . . . his preoccupation with heat extended to the subject of heat propagation in solid bodies, heat loss by radiation, and heat conservation."[1]

Fourier's investigations into heat culminated with the publication of the *Theorie Analytique de la Chaleur* [2]:

> There is no doubt that today this book stands as one of the most daring, innovative, and influential works of the nineteenth century on mathematical physics. . . . He worked with discontinuous functions when others dealt with continuous ones . . . and talked about the convergence of a series of functions before there was a definition of convergence. At the end of his 1811 prize essay, he even integrated 'functions' that have value ∞ at one point and are zero elsewhere. . . . It was the success of Fourier's work in applications that made necessary a redefinition of the concept of function, the introduction of a definition of convergence, . . . the ideas of uniform continuity and uniform convergence. It . . . was in the background of ideas leading to measure theory, and contained the germ of the theory of distributions.

1.4 Brownian Motion

The phenomenon of Brownian motion is named after the biologist Robert Brown who discovered it in 1827. Tiny particles suspended in a fluid make small irregular jumps, due to constant bombardment by the molecules of the fluid. The phenomenon thus provides evidence

[1] González-Velasco [10].
[2] *op. cit.*

for the molecular theory of matter. Albert Einstein showed in his paper in 1905 that the probability density function for Brownian motion satisfies the heat equation.

Brownian motion is obtained formally as a continuum limit of a *random walk* on a lattice. Consider a fair coin which comes down heads or tails with probability 1/2; and consider the process of flipping this coin successively. The tosses are independent, in the sense that the outcome of the n^{th} toss is independent of the outcomes of any of the preceding tosses. The process is represented as a sequence of independent random variables $\{X_n\}$ such that

$$\text{Prob } (X_n = \pm 1) = \frac{1}{2}.$$

The distribution function of a real valued random variable X is given by

$$F(x) = \text{Prob } (X < x).$$

The expectation $\mu = EX$ and variance $\sigma^2 = Var(X)$ of X are then defined by the Riemann-Stieltjes integrals

$$\mu = E(X) = \int_{-\infty}^{\infty} x dF(x), \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 dF(x).$$

By the definition of distribution function, F is non-decreasing in x. If F is differentiable, with $F' = p$, then $dF(x) = p(x)dx$.

The joint distribution function of two random variables X and Y is defined by

$$F(x, y) = \text{Prob } (X < x, Y < y).$$

The random variables X and Y are *independent* if

$$E(XY) = E(X)E(Y).$$

The random variables X_n corresponding to the n^{th} coin toss are independent and each has mean zero and variance 1. The coin tosses generate a random walk on the integers in which a particle is displaced to the right or left by one, according as the coin comes down head or tails. The position of the particle after n tosses is also a random variable,

$$S_n = \sum_{j=1}^{n} X_j. \tag{1.21}$$

Using the independence of the X_n, it is easily seen that S_n has mean zero and variance n. In fact,

$$E(S_n) = E(\sum_{j=1}^{n} X_j) = \sum_{j=1}^{n} E(X_j) = 0,$$

$$E(S_n^2) = E(\sum_{j,k=1}^{n} X_j X_k) = \sum_{j,k=1}^{n} E(X_j X_k) = \sum_{j=1}^{n} E(X_j^2) = n.$$

A formal passage to the continuum limit is obtained by measuring time steps in intervals of length Δt, and spatial steps of length Δx. Let $p(x,t)$ be the probability that the particle is at x at time t, assuming it began at the origin at $t = 0$. The particle can be at x at time $t + \Delta t$ only if it were at $x - \Delta x$ and moved to the right, or at $x + \Delta x$ and moved to the left, at time t. Therefore

$$p(x, t + \Delta t) = \frac{1}{2} \left(p(x - \Delta x, t) + p(x + \Delta x, t) \right). \tag{1.22}$$

We rewrite this equation as

$$\frac{p(x, t + \Delta t) - p(x,t)}{\Delta t} = \frac{\Delta x^2}{2\Delta t} \left(\frac{p(x - \Delta x, t) + p(x + \Delta x, t) - 2p(x,t)}{\Delta x^2} \right). \tag{1.23}$$

Now put

$$\sigma = \frac{\Delta x^2}{\Delta t},$$

and let Δx and Δt tend to zero with σ fixed. If we assume that in the limit $p(x,t)$ is a differentiable function, and take the limit of these difference quotients, we obtain

$$\frac{\partial p}{\partial t} = \frac{\sigma}{2} \frac{\partial^2 p}{\partial x^2},$$

i.e. the heat equation with $\sigma/2$ as the diffusion coefficient.

The constant σ is a free parameter in the mathematical theory which must be determined from physical principles. It was first determined by A. Einstein in his fundamental paper of 1905 [8] to be

$$\sigma = \frac{RT}{N} \frac{1}{6\pi k P},$$

where N is Avogadro's number, T is the temperature in degrees Kelvin, R is the universal constant in the gas equation, k is the coefficient of the viscosity of the fluid, and P is the radius of the particle (assumed spherical). The determination of a number of fundamental constants in the molecular theory of matter ha s been made from this formula for the diffusion constant and experimental observations.

The argument above is, of course, only formal, since one must pass from a probability function defined only at discrete points to a probablity density function which is defined and differentiable on all x and $t > 0$. It also does not tell us which solution of the heat equation we should take for p. As it turns out, p is precisely the fundamental solution of the heat equation. In fact, the fundamental solution of the heat equation is the probability density function is fundamentally related to the *Central Limit Theorem* of probability theory.

A random variable X is said to be *Gaussian* or *normal* if

$$\text{Prob } (X < b) = \int_{-\infty}^{b} \varphi(y)\, dy, \qquad \varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy. \tag{1.24}$$

The mean and variance of X are given by

$$E[X] = \int_{-\infty}^{\infty} y\varphi(y)\, dy = 0, \qquad E[X^2] = \int_{-\infty}^{\infty} y^2\varphi(y)\, dy = 1.$$

Theorem 1.4.1 [Central Limit Theorem]. *Let X_n be a sequence of independent, identically distributed random variables with finite expectation μ and variance σ^2: $\mu = E[X_n]$ and $\sigma^2 = E[(X_n - \mu)^2]$. Let $S_n = \sum_{j=1}^{n} X_n$. Then*

$$\lim_{n\to\infty} \text{Prob} \left(\frac{S_n - n\mu}{\sigma n^{1/2}} < b \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-y^2/2}\, dy.$$

In the case of the random walk on $\mathbb{Z}$, $\mu = 0,\ \sigma^2 = 1$, so the sequence of random variables

$$\frac{S_n}{\sqrt{n}}$$

tends to a normal random variable. We rescale the random walk by making the particle take jumps of size Δx at time intervals of length

Δt. Then the position of the particle at time $t = n\Delta t$ is given by $X_t = \Delta x S_n$. Therefore,

$$\frac{X_t}{\sqrt{t}} = \frac{\Delta x}{\sqrt{n\Delta t}} S_n = \sqrt{\sigma}\frac{S_n}{\sqrt{n}}, \qquad \sigma = \frac{(\Delta x)^2}{\Delta t}.$$

Letting $n \to \infty$, and Δx, $\Delta t \to 0$, with σ fixed, we see that

$$\frac{X_t}{\sqrt{\sigma t}}$$

is a Gaussian random variable. It is then a simple exercise to show that

$$\text{Prob } (X_t < x) = \frac{1}{\sqrt{2\pi\sigma t}} \int_{-\infty}^{x} e^{-y^2/2\sigma t} dy.$$

Hence the probability density function for Brownian motion satisfies the heat equation with diffusion coefficient $\sigma/2$.

We may think of a solution of the heat equation as a probability density function $p(x,t)$:

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/2\sigma t} p(y,0)\, dy$$

where $p(x,t)$ is the probability density of a particle moving randomly by a diffusion process, given that the initial probability density function of the particle is $p(x,0)$. Since p is a probability density, we should expect that $p(x,t) \geq 0$ for all $t > 0$ and that

$$\int_{-\infty}^{\infty} p(x,t)\, dx = 1$$

for all positive t. As we have already seen, these two properties are immediate consequences of the heat equation.

As a special case we may consider the situation where the particle is originally at the origin with ifs, ands, or buts, i.e. with probability 1. Its distribution function is then

$$F_X(x) = H(x) = \begin{cases} 0 & x < 0; \\ 1 & x \geq 0 \end{cases} \tag{1.25}$$

where H is the Heaviside step function. The density function δ of such a random variable must then have the property that

$$\int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx = 1$$

for any $\varepsilon > 0$. This property describes the *Dirac delta* function. The Dirac function is a generalized function; it is an example of a *distribution.* Such distributions play an important role in the modern theory of linear partial differential equations and will be discussed later.

The delta function is obtained as the distribution limit of the fundamental solution of the heat equation as $t \downarrow 0$:

$$\lim_{t \downarrow 0} \frac{e^{-x^2/2\sigma t}}{\sqrt{2\pi\sigma t}} = \delta(x).$$

Thus, the Dirac delta function is the "probability density" function of a particle which sits at the origin with probability 1, and the fundamental solution of the heat equation is the probability density function of this particle for $t > 0$ as it moves about randomly by diffusion.

We close with a few remarks about the Riemann-Stieltjes integral. For a more complete discussion, see [9], §25. Given a monotone non-decreasing right continuous function F, the integral

$$\int_a^b f(x) dF(x) \tag{1.26}$$

is defined as follows: Let $\mathcal{P}_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of the interval of $[a, b]$; and let

$$\Delta F_j = F(x_j) - F(x_{j-1}).$$

Given f be defined on $[a, b]$ and a partition $\mathcal{P}_n$ define

$$M_j = \sup_{x_{j-1} \le \xi \le x_j} f(\xi), \qquad m_j = \inf_{x_{j-1} \le \xi \le x_j} f(\xi).$$

We then define the upper and lower sums

$$U(\mathcal{P}_n, f, F) = \sum_{j=1}^{n} M_j \Delta F_j, \qquad L(\mathcal{P}_n, f, F) = \sum_{j=1}^{n} m_j \Delta F_j,$$

If

$$\sup_{\mathcal{P}_n} L_{a,b,f} = \inf_{\mathcal{P}_n} U_{a,b,f}$$

then f is said to be integrable with respect to F and the corresponding Riemann-Stieltjes integral (1.26) is defined to be this common value.

We leave it as an exercise to show that for any continous function f on the real line,

$$\int_{-\infty}^{\infty} f(x)dH(x) = f(0); \tag{1.27}$$

hence, formally, the derivative of the Heaviside step function is the Dirac delta function.

1.5 Exercises

1. Prove theorem 1.2.1. What is the probabilistic interpretation of the formula (1.12) for the fundamental solution of the heat equation in $\mathbb{R}^n$?

2. Find the explicit solution of the heat equation on the real line for initial data $u(x,0) = H(x)$, where $H(x)$ is the Heaviside step function (1.13).

3. Let both one-sided limits $f(x \pm 0) = \lim_{y \to x\pm} f(y)$ exist at a given point $x \in \mathbb{R}$, and let f be bounded and measureable on $\mathbb{R}$. Then
$$\lim_{t \downarrow 0} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\, dy = \frac{f(x+0) + f(x-0)}{2}.$$

4. Show that
$$u(x,t) = \frac{x}{t^{3/2}} e^{-x^2/4t}$$
satisfies the heat equation in $t > 0$. What is $\lim_{t \downarrow 0} u(x,t)$?

5. The total variation of a function f on the real line is given by the norm
$$||f||_v = \sup \sum_j |f(x_{j+1}) - f(x_j)|,$$
where the supremum is taken over all finite sequences $x_1 < x_2 < \cdots < x_n$. Show that the solution of the heat equation given by (1.3) satisfies
$$||u(\cdot,t)||_v \leq ||f||_v.$$
Moreover, show that if $f \in L^1(\mathbb{R})$ then $||u(\cdot,t)||_v$ tends to zero as $t \to \infty$, and give an estimate of the rate of decay.

6. Prove (1.19).

7. Compute the Fourier coefficients for the function (1.18).

8. Solve the boundary-initial value problem (1.15) when the boundary conditions on 0 and π are given by $u = 0$ and $u_x + hu = 0$

respectively, where h is a positive constant. Show the solution can be represented as a Fourier series

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\gamma_n^2 t} \sin \gamma_n x$$

where $\gamma_1 < \gamma_2 < \dots$ satisfy a transcendental equation. Show that the corresponding functions $\{\sin \gamma_n x\}$ are orthogonal on the interval $[0,\pi]$, and evaluate

$$\int_0^{\pi} \sin^2 \gamma_n x \, dx.$$

9. State and prove an analog of the maximum principle for the difference equation (1.23). What is the analog of fundamental solution for this difference equation?

10. Given any non-decreasing function $F(x)$, define functional $\Lambda_F(f)$ by
$$\Lambda_F(f) = -\int_{-\infty}^{\infty} f'(x)F(x)dx,$$
where f is continuously differentiable for $x \in \mathbb{R}$ and $f \to 0$ as $x \to \pm\infty$. Compute Λ_H, for the Heaviside step function H.

11. Compute
$$\int_0^1 x \, dF(x),$$
where F is the Cantor function.

Chapter 2

Laplace's Equation

2.1 Conservative vector fields and potentials

Let Ω be a bounded domain with a smooth boundary in $\mathbb{R}^n$, and consider the boundary-initial value problem for the heat equation

$$u_t = \Delta u, \quad x \in \Omega \qquad u(x,0) = u_0, \quad u\Big|_{\partial\Omega} = f.$$

As $t \to \infty$ the temperature will settle down (something we need to prove mathematically) to an equilibrium solution of the heat equation:

$$\Delta u = 0, \qquad u\Big|_{\partial\Omega} = f.$$

This is called the *Dirichlet* problem for Laplace's equation. Thus, Laplace's equation may be viewed as the equilibrium solution of the heat equation.

Laplace's equation also arises in potential theory, as follows. A vector field $\mathbf{F}$ is said to be conservative if the line integral

$$\int_x^a \mathbf{F} \cdot d\mathbf{y}$$

is independent of the path of integration. This line integral represents the work done in moving from x to a fixed reference point a (for example, to infinity). In this case we define the *potential energy* of the field to be the work done and we denote its value by Φ. It follows that

$$\nabla \Phi = -\mathbf{F}.$$

The scaler function Φ is called the *potential* of the vector field $\mathbf{F}$.

The best-known physical example of a conservative vector field is the inverse square law governing both the (classical, or Newtonian) gravitational field and the electric field. In both theories, the force in $\mathbb{R}^3$ between two point masses or charges is inversely proportional to the square of the distance between them. Thus, the force field due to a point source at the origin is given by

$$\mathbf{F}(\mathbf{x}) = -K\frac{\mathbf{x}}{|\mathbf{x}|^3}, \tag{2.1}$$

where K is a physical constant. We have chosen the minus sign here to represent a force of attraction. In the electrostatic case (known as Coulomb's law) like charges repel one another, and the minus sign is dropped. As the reader may easily verify, $\mathbf{F}$ is a conservative vector field for which the potential in $\mathbb{R}^3$ is

$$\Phi = \frac{K}{r}, \qquad r = |\mathbf{x}|.$$

Gauss' law of electrostatics states that for any (smoothly bounded) domain Ω,

$$\iint\limits_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S} = \text{sum of the charges enclosed in } \Omega. \tag{2.2}$$

For the case of a single point charge, the integral over the sphere B_ρ of radius ρ centered at the charge itself is easily computed. The outward unit normal is $\nu = \mathbf{x}/|\mathbf{x}|$, so in spherical coordinates

$$\iint\limits_{B_\rho} \frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot d\mathbf{S} = \iint\limits_{B_\rho} \frac{\mathbf{x}}{\rho^3}\cdot\frac{\mathbf{x}}{\rho}\,dS = \int_0^\pi\int_0^{2\pi}\frac{1}{\rho^2}\rho^2\sin\theta\, d\theta\, d\varphi = 4\pi.$$

As a simple computation shows, the divergence of the vector field (2.1) is zero in the region excluding the origin; hence the integral (2.2) over any smoothly bounded region containing the point source is independent of the region. This result follows from the Gauss divergence theorem,

$$\iiint\limits_{A} \operatorname{div}\mathbf{F}\,dx = \iint\limits_{\partial A} \mathbf{F}\cdot d\mathbf{S},$$

for any smoothly bounded region A in which $\mathbf{F}$ is continuously differentiable. Simply let A be the region $\Omega \setminus B_\rho$, where ρ is chosen so that $B_\rho \subset \Omega$.

The two equations

$$\mathbf{F} = -\nabla\,\Phi, \qquad \operatorname{div}\mathbf{F} = 0$$

lead to Laplace's equation for Φ:

$$\Delta\Phi = 0.$$

Solutions of Laplace's equation are called *harmonic functions* and possess a number of very strong mathematical properties, which we shall discuss in this chapter.

The function

$$\frac{1}{4\pi r} \tag{2.3}$$

is the fundamental solution of Laplace's equation in $\mathbb{R}^3$. An easy calculation (e.g. in spherical coordinates) shows that $\Delta\Phi = 0$ everywhere except at the origin. Its gradient is the vector field $\mathbf{F}$, the force field due to a point charge.

The potential due to a sum of point charges at the points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is

$$\sum_{j=1}^{n} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_j|};$$

and, by analogy, the potential due to a continuous distribution of sources with density ρ is

$$\Phi(x) = \iiint\limits_{\Omega} \frac{\rho(y)}{4\pi|x - y|} dy.$$

We shall prove that Φ satisfies Poisson's equation, $\Delta\Phi = -\rho$.

Theorem 2.1.1 *Let Ω be a smoothly bounded finite domain in $\mathbb{R}^3$, and let u be a C^2 function in Ω. Then we have the fundamental identity*

$$u(x) + \iiint\limits_{\Omega} \frac{\Delta u}{4\pi r}\, dy + \iint\limits_{\partial\Omega} \left(u \frac{\partial}{\partial\nu} \frac{1}{4\pi r} - \frac{1}{4\pi r} \frac{\partial u}{\partial \nu} \right) dS_y = 0, \tag{2.4}$$

where $r = |x - y|$, ν is the outward unit normal and dS_y the element of surface area on the boundary of Ω.

Proof: We begin by stating Green's identity of multi-variable calculus:

$$\iiint_{\Omega} u\Delta v - v\Delta u \, dx = \iint_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS, \tag{2.5}$$

where Ω is any smoothly bounded domain in R^3, ν denotes the outward unit normal to the surface $\partial\Omega$, and u and v are continuously differentiable in the interior of Ω. This identity is a direct consequence of the Gauss divergence theorem.

Now let $v = 1/4\pi|x-y|$, B_δ be the sphere of radius δ centered at x, and $\Omega_\delta = \Omega \setminus B_\delta$. Both u and v being smooth inside Ω_δ, we can apply Green's identity. Since $\Delta_y v = 0$, $y \neq x$ (Δ_y denotes the Laplacian with respect to the y variables), we have

$$-\iiint_{\Omega_\delta} \frac{\Delta u}{4\pi|x-y|} dy = \iint_{\partial\Omega_\delta} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS_y.$$

As $\delta \to 0$, the integral on the left tends to $-\int_\Omega \Delta u/4\pi|x-y|\, dy$, since $1/r$ is locally integrable in R^3 and Δu is continuous. In the surface integral over ∂B_δ, choose spherical coordinates centered at x. Then

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial B_\delta} = -\left.\frac{\partial v}{\partial r}\right|_{r=\delta} = \frac{1}{4\pi\delta^2}.$$

(Recall that ν is the outward unit normal from the interior of Ω_δ, hence the minus sign.) Hence

$$\iiint_{\partial B_\delta} u\frac{\partial v}{\partial \nu}\, dS_y = \frac{1}{4\pi\delta^2} \iint_{\partial B_\delta} u \, dS_y.$$

On ∂B_δ, $dS_y = \delta^2 \sin\theta \, d\theta \, d\varphi$. Since u is continuous in Ω, the surface integral over ∂B_δ tends to $u(x)$ as $\delta \to 0$.

On the other hand, the integral of $v\partial u/\partial\nu$ over B_δ tends to zero with δ, since on that surface $\partial u/\partial \nu$ is bounded while $v = O(\delta^{-1})$. Letting $\delta \to 0$ we obtain (2.4). ■

The above calculation goes through if $1/4\pi r$ is replaced by a more general kernel

$$G(x,y) = \frac{1}{4\pi|x-y|} + h(x,y),$$

where h is harmonic in y for each x. If h can be constructed so that $G(x,y)=0$ for each $x \in \Omega$, and $y \in \partial\Omega$, then the representation (2.4) would simplify to

$$u(x) = -\iiint\limits_{\Omega} G(x,y)\Delta u\, dy - \iint\limits_{\partial\Omega} u \frac{\partial G}{\partial \nu_y}\, dS_y; \tag{2.6}$$

In that case the solution of the boundary value problem

$$\Delta u = -\rho, \qquad x \in \Omega; \qquad u\Big|_{\partial\Omega} = g,$$

would be given by

$$u(x) = \iiint\limits_{\Omega} G(x,y)\rho(y)\, dy - \iint\limits_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu_y}\, dS_y$$

In particular, if u is harmonic in the interior of Ω and takes values g on the boundary, then

$$u(x) = -\iint\limits_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu_y} dS_y.$$

The function G is called the *Green's function* for the Dirichlet problem (on the domain Ω). The Green's function $G(x,y)$ is harmonic in y and vanishes for $y \in \partial\Omega$.

Theorem 2.1.2 *The Green's function for a smoothly bounded domain Ω is symmetric in x and y: $G(x,y) = G(y,x)$. It follows that the Green's function is also harmonic in x and vanishes for $x \in \partial\Omega$.*

Proof: Letting $x \neq x' \in \Omega$, we apply Green's identity to the functions $u(y) = G(x,y)$ and $v(y) = G(x',y)$. Let B_δ and B'_δ be spheres of radius δ about the points x and x', with δ small enough that both spheres lie in Ω and do not intersect. Since u and v are both harmonic in $\Omega_\delta = \Omega \setminus (B_\delta \cup B'_\delta)$, we get

$$\iint\limits_{\partial\Omega_\delta} \left(G(x,y)\frac{\partial G(x',y)}{\partial \nu_y} - G(x',y)\frac{\partial G(x,y)}{\partial \nu_y} \right) dS_y = 0.$$

Since $G(x,y)$ and $G(x',y)$ both vanish on $\partial\Omega$, the integral above reduces to an integral over $\partial B_\delta \cup \partial B'_\delta$. Letting $\delta \to 0$ and proceeding as in the proof of Theorem 2.1.1, we obtain $G(x,x') - G(x',x) = 0$. ■

The proof of the existence of a Green's function for a general domain is somewhat technical, and we won't go into that here. However, in domains with a high degree of symmetry the Green's function can often be constructed explicitly. For example, the Green's function for the half space $\mathbb{R}^3_+ = \{x : x_3 > 0\}$ can be constructed by the *method of images.* Given a source at y we place an image at $y' = (y_1, y_2, -y_3)$. Then for $x_3 > 0$ the function

$$\frac{1}{|x-y'|} = \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3+y_3)^2}}$$

is a harmonic function of y in $y_3 > 0$. The Green's function for $\mathbb{R}^3_+$ is then

$$G(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|x-y'|}\right). \tag{2.7}$$

This function has a fundamental singularity at $x = y$ and vanishes on the plane $x_3 = 0$. We obtain a representation of the solution of the Dirichlet problem by calculating the outward normal derivative of G on $x_3 = 0$. By (2.6), if u is harmonic in $x_3 > 0$ and $u = f$ on $x_3 = 0$, then

$$u(x) = \frac{x_3}{2\pi}\iint_{\mathbb{R}^2} \frac{f(y_1,y_2)\,dy_1\,dy_2}{\left((x_1-y_1)^2+(x_2-y_2)^2+x_3^2\right)^{3/2}} \tag{2.8}$$

The solution is thus given by a convolution on $x_3 = 0$ of the boundary data f with the kernel

$$K(x,z) = \frac{z}{2\pi}\frac{1}{(|x|^2+z^2)^{3/2}}, \qquad x \in \mathbb{R}^2, \quad z > 0.$$

A reflection method also works to obtain the Green's function for balls. If B is a ball of radius R centered at the origin, and $y \in B$, then the reflection of y in ∂B is the point

$$y' = \frac{R^2}{|y|^2}y.$$

We leave it as an exercise to show that the Green's function for the sphere of radius R is

$$G(x,y) = \frac{1}{4\pi|x-y|} - \frac{R}{|y|}\frac{1}{4\pi|x-y'|}. \tag{2.9}$$

If u is harmonic inside a sphere of radius R and continuous onto the sphere, then in the interior u is given by the Poisson integral

$$u(x) = \frac{R^2 - r^2}{4\pi R}\iint\limits_{S_R} \frac{u(y)}{|x-y|^3} dS_y, \qquad r^2 = |x|^2. \tag{2.10}$$

Three fundamental properties of harmonic functions follow immediately from this formula:

Theorem 2.1.3 *Let u be harmonic in a domain Ω. Then at every $x \in \Omega$, u is equal to its average value on any sphere $B_\rho(x)$ centered at x and contained in Ω:*

$$u(x) = \frac{1}{4\pi\rho^2}\iint\limits_{|y-x|=\rho} u(y)\rho^2 d\omega, \qquad d\omega = \sin\theta \, d\theta \, d\varphi,$$

This is the mean value theorem for harmonic functions.

The *strong maximum principle* for harmonic functions follows immediately from the mean value theorem.

Theorem 2.1.4 *If u is harmonic in a domain Ω and u attains an interior maximum or minimum in Ω, then u is identically constant.*

Proof: Suppose $m \le u \le M$ in Ω and $u = M$ at an interior point $x \in \Omega$, then choose a small sphere centered at x and contained in Ω. If u were strictly less than M anywhere on that sphere, the average value of u over that sphere would be strictly less than M (u is continuous); and the mean value property would be violated. Hence u must be identically equal to M everywhere on that sphere, and so $u \equiv M$ on the largest sphere containing x and lying in Ω.

Now propagate this extreme value of u throughout Ω, as follows. Let $M_u = \{x : x \in \Omega, \ u(x) = M\}$. Clearly M_u is closed, since u is continuous; but M_u is also open by the preceding argument. Therefore, M_u is either empty or the entire set Ω. ■

Finally, we obtain *Harnack's inequality* from (2.10).

Theorem 2.1.5 *Let u be a non-negative harmonic function in a ball B_R of radius R. Then for x in B_R*

$$\frac{R}{R+|x|}\frac{R-|x|}{R+|x|}u_0 \le u(x) \le \frac{R}{R-|x|}\frac{R+|x|}{R-|x|}u_0,$$

where u_0 is the value of u at the center of the sphere.

Proof: Choose coordinates so that the sphere is centered at the origin. For any $|x| = r < R$ and all y, $|y| = R$ we have

$$\frac{1}{(R+r)^3} \le \frac{1}{|x-y|^3} \le \frac{1}{(R-r)^3}.$$

The Poisson kernel therefore satisfies the inequality

$$\frac{1}{4\pi R}\frac{R-|x|}{R+|x|}\frac{1}{R+|x|} \le \frac{R^2-|x|^2}{4\pi R}\frac{1}{|x-y|^3} \le \frac{1}{4\pi R}\frac{R+|x|}{R-|x|}\frac{1}{R-|x|}.$$

Multiplying this inequality through by $u(y) > 0$, integrating over the sphere $|y| = R$, and applying the mean value theorem, we obtain the result. ■

All three of these theorems are valid in any dimension $n \ge 2$. The maximum principle is valid for general second order uniformly elliptic scalar partial differential equations with continuous coefficients. The Harnack inequality can also be extended to more general second order elliptic equations, but not in the generality of the case of constant coefficients. (see Moser [13] and Serrin [20]).

A number of properties of harmonic functions follow from Harnack's inequality. These are left as exercises.

Let us prove the converse of the mean value theorem:

Theorem 2.1.6 *Let u be a continuous function which satisfies the mean value property on every sphere S contained in a domain Ω. Then u is harmonic in Ω.*

Proof: Let S be any sphere contained in Ω. Using the Poisson integral, we construct a harmonic function v with $v = u$ on ∂S. Both u and v satisfy the mean value property in the interior of S, hence $w = u - v$ satisfies the mean value property. Therefore w satisfies the maximum principle inside S, and w vanishes on the boundary. Therefore $w \equiv 0$ inside S, and u is harmonic inside S. This argument clearly applies anywhere in Ω, so u is harmonic in Ω. ■

2.2 Polar and Spherical coordinates

The Laplacian in polar coordinates is

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right]=0. \tag{2.11}$$

We look for a solution of this equation by the method of separation of variables; thus we look for a solution of the form $u(r,\theta) = X(r)\Theta(\theta)$. Substituting this into Laplace's equation and repeating the same procedure we used for the heat equation, we obtain the pair of equations

$$\frac{r}{X}(rX')' = -\frac{\Theta''}{\Theta} = \gamma^2;$$

hence

$$r^2X'' + rX' - \gamma^2 X = 0, \qquad \Theta'' + \gamma^2\Theta = 0.$$

Since Θ should be 2π periodic in θ, we must take γ to be an integer n. We then obtain the solutions $X_n = r^{\pm n}$ and $\Theta_n = \cos n\theta$, $\sin n\theta$, and the two sets of solutions

$$r^{\pm n}(a_n \cos n\theta + b_n \sin n\theta), \qquad n \geq 0.$$

The set of solutions with negative powers of r is regular at infinity and is used when solving the Dirichlet problem in the exterior of a disk.

As in the case of the heat equation, we look for a general solution of the boundary value problem $\Delta u = 0,\ 0 \leq r \leq R;\ u(R,\theta) = f(\theta)$, by the method of superposition, that is,

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^n.$$

The trigonometric functions satisfy the orthogonality conditions

$$\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0,$$

$$\int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \delta_{nm}\pi.$$

The boundary condition $u(R,\theta) = f(\theta)$ leads to a determination of the Fourier coefficients

$$a_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \qquad b_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Given these values of the Fourier coefficients, we may write u as an integral

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\theta')\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\theta')\right] d\theta'.$$

This series can be summed to give

$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi} \frac{f(\theta')}{r^2+R^2-2rR\cos(\theta-\theta')}\, d\theta'. \tag{2.12}$$

Spherical coordinates in $\mathbb{R}^3$ are given by

$$x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi \qquad x_3 = r\cos\theta$$

and the Laplacian in these coordinates is given by

$$\Delta u = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \left(\frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\varphi^2}\right)\right]. \tag{2.13}$$

Consider the expansion of the Coulomb potential $1/|x-y|$ when the coordinate system is chosen so that y lies on the x_3 axis, i.e. $y = (0,0,R)$. Then

$$\frac{1}{|x-y|} = \frac{1}{\sqrt{r^2+R^2-2rR\cos\theta}} = \begin{cases} \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n P_n(\cos\theta) & r < R; \\ \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n P_{-n}(\cos\theta) & r > R. \end{cases} \tag{2.14}$$

The functions $P_{\pm n}$ are polynomials of degree n in $\cos\theta$. The polynomials P_n, $n \geq 0$ are called the *Legendre* polynomials. They satisfy a second order ordinary differential equation which may be obtained by substituting (2.14) into Laplace's equation. We shall see that the polynomials P_{-n} are multiples of P_{n-1}.

The function $|x-y|^{-1}$ is a harmonic function of x for fixed y, hence the sums in (2.14) are also harmonic functions of x. Substituting the expression $r^n P_n(\cos\theta)$ into Laplace's equation in spherical coordinates, we obtain the ordinary differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n}{d\theta}\right)+n(n+1)P_n=0. \tag{2.15}$$

Applying the same argument to $r^{-n}P_n(\cos\theta)$ instead, we find that P_{-n} satisfies the same equation with n replaced by $n-1$.

Under the change of variable $z=\cos\theta$ this differential equation is transformed into

$$\frac{d}{dz}\left((1-z^2)\frac{dP_n}{dz}\right)+n(n+1)P_n=0. \tag{2.16}$$

This equation, called Legendre's equation, is a special case of the *hypergeometric equation*, cf. Whittaker and Watson [27].

The spherical Laplacian is the Laplacian defined on functions restricted to the sphere, that is, functions which are independent of r. It is therefore the differential operator

$$\Delta u=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}\right)\right].$$

The eigenfunctions of the Laplacian on the sphere are non-trivial solutions of the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\lambda Y+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}=0.$$

The Legendre polynomials $P_n(\cos\,\theta)$ constructed above are one set of solutions of this equation, with eigenvalue $\lambda_n=n(n+1)$. These eigenfunctions are *axisymmetric*, that is, they are invariant under rotations about the $z-$axis. We look for more general eigenfunctions of the Laplacian of the form $Y(\theta,\varphi)=P_n^m(\cos\theta)e^{im\varphi}$ where m is an integer. With the substitution $z=\cos\theta$ that we used before, we obtain the differential equation

$$\frac{d}{dz}\left((1-z^2)\frac{dP_n^m}{dz}\right)+\left(\lambda_{n,m}-\frac{m^2}{1-z^2}\right)P_n^m=0. \tag{2.17}$$

The eigenfunctions $P_n^m(\cos\theta)e^{im\varphi}$ are known as the *spherical harmonics*, and the polynomials P_n^m are called the *associated Legendre polynomials*. It turns out that the associated eigenvalues $\lambda_{m,n}$ are independent of m, and are given by $n(n+1)$. In other words, the eigenvalues $n(n+1)$ are *degenerate*; they are of multiplicity $2n+1$, corresponding to the $2n+1$ independent functions $e^{im\varphi}$, $m = -n, \ldots, n$.

Just as the functions $\{e^{in\theta}\}$ form a complete orthogonal sequence of functions for $L^2(-\pi,\pi)$, the spherical harmonics form a complete orthogonal basis for the Hilbert space $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$.

2.3 Analytic function theory.

A function $f = f(z)$, where $z \in C$ is a complex variable, is said to be *analytic* in a domain $\Omega \in C$ if it is differentiable, that is, if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists for $z_0 \in \Omega$. By definition, this limit must be the same along any path leading to z_0; in particular, we must get the same values for the derivative as $z \to z_0$ along the paths $\Re z = \Re z_0$ and $\Im z = \Im z_0$. Writing $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$ this constraint leads to the *Cauchy Riemann* equations

$$u_x = v_y, \qquad v_x = -u_y. \tag{2.18}$$

This is an *elliptic* system of partial differential equations. One may show readily that both u and v are harmonic.

We develop some of the basic properties of analytic functions from the viewpoint of elliptic partial differential equations. We begin by introducing the variable $\bar{z} = x - iy$. We treat z, $\bar{z}$ as independent coordinates in the real plane. By the chain rule,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

and the Cauchy Riemann equations can be written in the compact form

$$\frac{\partial f}{\partial \bar{z}} = 0, \qquad f(z,\bar{z}) = u(z,\bar{z}) + iv(z,\bar{z}).$$

The first order partial differential operator $\partial/\partial\bar{z}$ is known as the $\bar{\partial}$ operator. We will show that its fundamental solution is

$$\frac{1}{2\pi i z}$$

A general function in the plane can be written as a function of the two variables z and $\bar{z}$. Suppose that $\bar{\partial} f = \mu$. We may write the Green-Stokes theorem in complex coordinates:

$$\int_{\partial\Omega} h\,dt = \iint_{\Omega} d(h\,dt) = \iint_{\Omega} \frac{\partial h}{\partial \bar{t}} d\bar{t} \wedge dt.$$

If $t = x + iy,\ \bar{t} = x - iy$, then

$$d\bar{t} \wedge dt = (dx - idy) \wedge (dx + idy) = 2idx \wedge dy.$$

Now

$$d\frac{f(t,\bar{t})\,dt}{2\pi i(t-z)} = \frac{\partial}{\partial \bar{t}} \frac{f(t,\bar{t})}{2\pi i(t-z)} d\bar{t} \wedge dt.$$

Let $\Omega_\varepsilon = \Omega \setminus D_\varepsilon$ where D_ε is the disk of radius ε centered at $z \in \Omega$. Since $1/(t-z)$ is an analytic function of t for $t \neq z$,

$$\frac{\partial}{\partial \bar{t}} \frac{f(t,\bar{t})}{2\pi i(t-z)} = \frac{\mu(t,\bar{t})}{2\pi i(t-z)}, \qquad t \in \Omega_\varepsilon.$$

We orient the boundary of Ω_ε so that the interior of Ω_ε lies to the left of the contour of integration. Then

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(t,\bar{t})}{(t-z)} dt - \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(t,\bar{t})}{t-z} dt = \frac{1}{2\pi i} \iint_{\Omega_\varepsilon} \frac{\mu}{(t-z)} d\bar{t} \wedge dt,$$

where ∂D_ε is oriented in the counterclockwise direction. Letting $t = z + \varepsilon e^{i\theta}$, $dt = i\varepsilon e^{i\theta} d\theta$, we have

$$\frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(t,\bar{t})}{(t-z)} dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta}, \bar{z} + \varepsilon e^{-i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta \to f(z,\bar{z}),$$

as $\varepsilon \to 0$.

We leave it to the reader to show that

$$\lim_{\varepsilon \to 0} \iint\limits_{\Omega_\varepsilon} \frac{\mu}{t-z} d\bar{t} \wedge dt = \iint\limits_{\Omega} \frac{\mu}{t-z} d\bar{t} \wedge dt.$$

We thus obtain

$$f(z, \bar{z}) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(t, \bar{t})}{t-z} dt - \frac{1}{2\pi i} \iint\limits_{\Omega} \frac{\bar{\partial} f}{t-z} d\bar{t} \wedge dt.$$

In particular, if f is analytic inside Ω then we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(t)}{t-z} dt.$$

This is the analog of the Poisson integral for harmonic functions. It says that a function analytic in a domain Ω is the convolution of its boundary values with the fundamental solution of the $\bar{\partial}$ operator on the bound ary of Ω.

Potential theory is used as a simple model of irrotational, incompressible inviscid fluid flow. If $\vec{v}$ denotes the velocity of a fluid, these two conditions are expressed mathematically by the equations

$$\begin{aligned} \operatorname{curl} \vec{v} &= 0, \qquad && \text{irrotational} \\ \operatorname{div} \vec{v} &= 0, && \text{incompressible.} \end{aligned}$$

If the domain is simply connected, then $\operatorname{curl} \vec{v} = 0$ implies that $\vec{v}$ is the gradient of a scalar function φ. Then

$$\operatorname{div} \vec{v} = \operatorname{div} \nabla \varphi = 0,$$

hence φ is harmonic. φ is called the *velocity potential.*

Since the fluid is inviscid, the only boundary condition to be satisfied is that the normal velocity of the fluid vanishes on any solid obstacle. Thus, φ satisfies the boundary value problem

$$\Delta \varphi = 0, \quad x \in \Omega; \qquad \left. \frac{\partial \varphi}{\partial \nu} \right|_{\partial\Omega} = 0.$$

Such a problem is called a *Neumann* problem.

In two space dimensions, if φ is harmonic, we may construct a second harmonic function ψ from the Cauchy Riemann equations: $\psi_x = \varphi_y$, $\psi_y = -\varphi_x$. These equations form a set of overdetermined equations for ψ; for we cannot impose the first partial derivatives of a function ψ arbitrarily: we must have $\psi_{xy} = \psi_{yx}$. This leads to the integrability condition $\varphi_{xx} + +\varphi_{yy} = 0$, which is satisfied automatically since φ is harmonic. The function ψ is single valued if the domain is simply connected, since

$$\oint d\psi = \oint \psi_x \, dx + \psi_y \, dy = \oint \varphi_y \, dx - \varphi_x \, dy = -\iint \Delta\varphi \, dxdy = 0.$$

The function ψ, the conjugate harmonic function, is called the *stream function.* Its gradient is orthogonal to the velocity field; and its level curves are the *streamlines* of the flow, that is, the trajectories of the fluid particles.

The function $f = \varphi + i\psi$ is an analytic function of $z = x + iy$, since the Cauchy Riemann equations are satisfied. If we write the fluid velocity as a complex quantity, $v = v_1 + iv_2$, then $v = \overline{f'(z)}$.

Complex variable methods can be used to construct solutions of the Dirichlet or Neumann problem in a variety of domains in the plane by the method of *conformal mapping.* We leave it as an exercise to show that the stream function must be constant on the boundary an obstacle in the stream. Hence a flow problem can be expressed as a Dirichlet problem for the stream function.

As an example, consider the problem of finding the flow of an incompressible, irrotational fluid in the exterior of the unit disk $|z| \leq 1$.

We begin by noting that the potential φ for the uniform flow $\vec{v} = (U, 0)$ is given by $\varphi = Ux$. The stream function is then given by $\psi = Uy$ and the complex velocity potential is $f(z) = Uz$. We leave it as an exercise to show that the conformal transformation

$$z \mapsto w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{2.19}$$

maps the exterior of the unit disk in the z-plane one-to-one and onto the w-plane. It follows that the potential

$$f(z) = U\left(z + \frac{1}{z}\right) \tag{2.20}$$

is analytic in the exterior of the unit disk in the z plane and is asymptotic to Uz at infinity. The stream function, $\psi = \Im f(z)$, is easily seen to be constant on the unit circle:

$$\psi(1, \theta) = \Im U(e^{i\theta} + e^{-i\theta}) = \Im\, 2U \cos\theta = 0.$$

The force on an obstacle B due to the flow around it is given by the line integral

$$\vec{f} = -\oint_{\partial B} p\hat{\nu}\, ds,$$

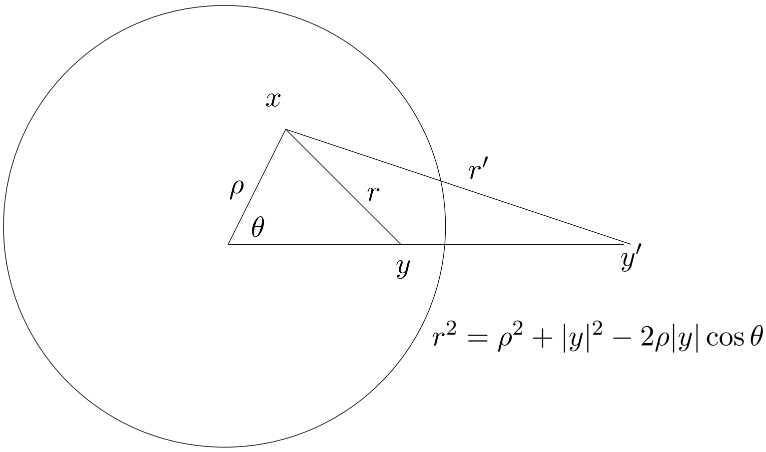
where p is the hydrodynamic pressure, and $\hat{\nu}$ is the outward normal to B. The pressure is not obtained directly in this simple theory. Later we shall prove Bernouilli's law, namely that

$$\frac{1}{2}v^2 + p$$

is constant along streamlines, where v is the speed of the fluid, in this case $|f'(z)|^2$. This makes it possible to calculate the lift and drag on the unit circle due to the flow (2.20).

2.4 Exercises

1. Show that (2.9) is the Green's function for the sphere in $\mathbb{R}^3$ of radius R: Show it is harmonic in y and vanishes on $|y| = R$. Find the Poisson integral (2.10) for the sphere of radius R in $\mathbb{R}^3$. That is, calculate the outward normal derivative of (2.9) on the sphere $|y| = R$. (Use the diagram below.)



2. Prove Weierstrass' convergence theorem: If u_n is a sequence of harmonic functions continuous on the closure of a domain Ω, and if u_n converges uniformly on $\partial\Omega$, then u_n converges uniformly to a harmonic function u in the interior of Ω.

3. Prove: A harmonic function which is non-negative in $\mathbb{R}^n$ is constant.

4. If $\{u_n\}$ is a monotone sequence of harmonic functions on a domain Ω, and if $\{u_n(P)\}$ converges for some fixed $P \in \Omega$, then $\{u_n\}$ converges uniformly on every compact subset of Ω; and the limit function is itself harmonic.

5. If $\{u_n\}$ is a uniformly bounded sequence of harmonic functions on Ω then the partial derivatives $\partial u/\partial x_j$ are uniformly bounded on compact subdomains of Ω. Hence $\{u_n\}$ contains a subsequence that converges uniformly on compact subdomains of Ω.

6. Prove that the Legendre polynomials $P_n(z)$ are orthogonal for $n \neq n'$ on $(-1, 1)$. Show that this orthogonality condition, plus

the fact that $P_n(z)$ is a polynomial of degree n, determines the Legendre polynomials up to a multiplicative constant.

7. Prove that the stream function for an incompressible, irrotational flow is constant on a fixed boundary. Using Bernouilli's theorem, calculate the lift and drag on the unit circle due to the flow (2.20).

8. It is possible, in a multiply connected domain, for a velocity field to be irrotational but not have a single-valued velocity potential. Show that the function

$$f(z) = -U\left(z + \frac{1}{z}\right) + \frac{\Gamma}{2\pi i}\log z, \qquad \Gamma \text{ real}, \tag{2.21}$$

is multiple-valued and analytic in the exterior of the unit disk, but that it defines a single-valued velocity field. Find the stream function and show that $f(z)$ defines a flow in the exterior of the unit disk; that is, that the stream function is constant on the unit circle. Calculate the circulation for this flow. What is the asymptotic velocity at infinity? What is the net force on the unit disk exerted by this flow?

9. A function u is said to be subharmonic in Ω if $\Delta u \geq 0$. Show that a subharmonic function cannot attain an interior maximum. Show that if u is harmonic, then $|\nabla u|^2$ is subharmonic.

Chapter 3

The Wave Equation

3.1 The wave equation in 1,2,3 dimensions

The initial value problem for the wave equation in n space dimensions is given by

$$\Box\, u = 0, \quad u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x). \tag{3.1}$$

where $\Box$ denotes the *D'Alembertian* $\Box\, u = u_{tt} - c^2 \Delta\, u$.

The solution of (3.1) in $\mathbb{R}^1$ is due to D'Alembert. It is apparent that the one dimensional wave equation is satisfied by any function of the form $f(x+ct) + g(x-ct)$. Any function of the variable $x - ct$ will appear as a wave-form travelling to the right with speed c; while any function of $x + ct$ will appear as a wave travelling to the left with speed c. Thus the solution of the wave equation appears as a superposition of waves travelling to the left and right with speed c.

The initial conditions lead to the equations

$$f + g = u_0, \qquad f' - g' = \frac{1}{c} u_1.$$

We solve this pair of equations to obtain

$$f' = \frac{1}{2}\left(u_0' + \frac{1}{c}u_1\right), \qquad g' = \frac{1}{2}\left(u_0' - \frac{1}{c}u_1\right).$$

As solutions to this pair of equations we take

$$f(x) = \frac{1}{2}u_0(x) + \frac{1}{c}\int_0^x u_1(y)\,dy, \qquad g(x) = \frac{1}{2}u_0(x) - \frac{1}{c}\int_0^x u_1(y)\,dy.$$

We then obtain *D'Alembert's* solution of the wave equation in one space dimension:

$$u(x,t) = \frac{u_0(x+ct)+u_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(y)\,dy. \tag{3.2}$$

We see that the solution at (x,t) depends only on the values of u_0 at $x \pm ct$ and on the values of $u_t(x,0)$ on the interval $(x-ct, x+ct)$. The region in the triangle in Figure 3.1 is called the *domain of dependence* of the point (x,t).

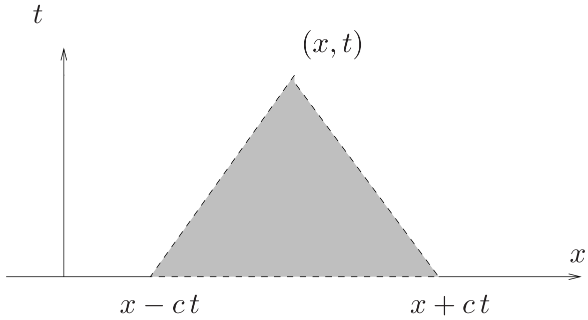


Figure 3.1: Domain of Dependence

The fundamental solution for the wave equation in $\mathbb{R}^n$ is defined to be the solution W_n of the wave equation which satisfies the initial value problem

$$\Box W_n = 0, \quad W_n(x,0) = 0, \quad \frac{\partial W_n}{\partial t}(x,0) = \delta(x).$$

Equivalently, the function $v = W_n * f$ satisfies

$$\Box v = 0, \qquad v(x,0) = 0, \qquad v_t(x,0) = f(x). \tag{3.3}$$

From (3.2) we see that

$$W_1(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(y)\,dy = \frac{H(x+ct)-H(x-ct)}{2c},$$

where H is the Heaviside step function

$$H(s) = \begin{cases} 1 & s > 0 \\ \frac{1}{2} & s = 0 \\ 0 & s < 0 \end{cases}$$

We leave it as an exercise to show that for $t > 0$, W_1 can be written in the compact form

$$W_1(x,t) = \frac{H(c^2t^2 - x^2)}{2c}, \qquad t > 0. \tag{3.4}$$

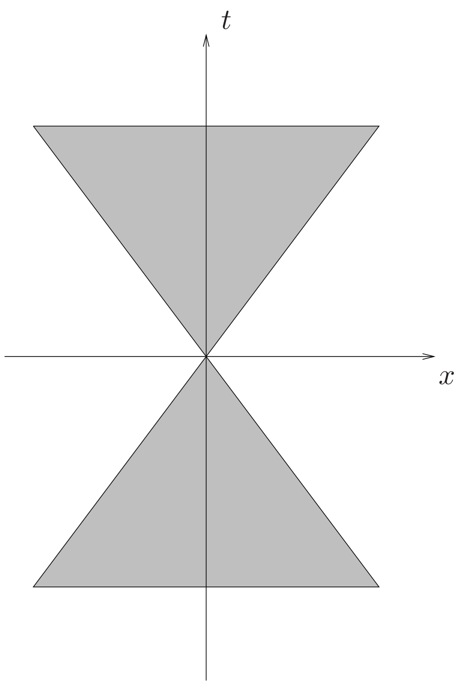


Figure 3.2: Forward and Backward Light Cone

The shaded area in Figure (3.2) is called the *domain of influence* of the pulse. It extends forward and backward in time since the wave equation is invariant under time reversal. Unlike the heat equation, disturbances propagate at a finite speed.

Since the wave equation has constant coefficients, we should expect that $\partial W_1/\partial t$ is also a solution of the wave equation. We have already observed that the distributional derivative of the Heaviside function is the Dirac delta function. We leave it as an exercise to verify that

$$\frac{\partial W_1}{\partial t}(x,0) = \frac{\delta(x+ct)+\delta(x-ct)}{2}, \tag{3.5}$$

and that the solution of (3.1) in $\mathbb{R}^1$ can be written as $u = W_1 * u_1 + W_{1,t} * u_0$.

A representation of the explicit solution in $\mathbb{R}^3$ was obtained by Kirchoff, using the method of spherical means. We verify Kirchoff's solution here; and later, in §3.4, we present a method for finding it deductively. The spherical mean of a function f in $\mathbb{R}^3$ is given by

$$M_f(x,t) = \frac{1}{4\pi c^2 t^2} \iint\limits_{|x-y|=ct} f(y)\, dS_y.$$

For any $f \in C^2(\mathbb{R}^3)$ consider the function

$$u(x,t) = 4\pi t M_f = \frac{1}{c^2 t} \iint\limits_{|y-x|=ct} f(y)\, dS_y.$$

We write $y = x + ct\omega$ and $dS_y = c^2 t^2 d\omega$, where ω is a vector on the unit sphere and $d\omega$ denotes the element of surface area on the unit sphere. Then

$$u(x,t) = t \iint\limits_{|\omega|=1} f(x+ct\omega)\, d\omega, \tag{3.6}$$

and

$$\frac{\partial u}{\partial t} = \iint\limits_{|\omega|=1} f(x+ct\omega)\, d\omega + ct \iint\limits_{|\omega|=1} \sum_{j=1}^{3} \frac{\partial f}{\partial y_j}\omega_j\, d\omega = \frac{1}{t}\left(u + cW\right),$$

where

$$W = t^2 \iint\limits_{|\omega|=1} \sum_{j=1}^{3} \frac{\partial f}{\partial y_j}\omega_j\, d\omega = \frac{1}{c^2} \iint\limits_{|y-x|=t} \frac{\partial f}{\partial \nu}\, dS_y.$$

It then follows that

$$\frac{\partial^2 u}{\partial t^2} = \frac{c}{t}\frac{\partial W}{\partial t}.$$

By the divergence theorem,

$$W = \frac{1}{c^2} \iiint\limits_{|x-y|<ct} \Delta_y f(y) dy = \frac{1}{c^2} \int_0^{ct} \iint\limits_{|\omega|=1} \Delta_x f(x+\rho\omega)\, d\omega\, \rho^2\, d\rho$$

$$= \Delta_x \frac{1}{c^2} \int_0^{ct} \iint\limits_{|\omega|=1} f(x+\omega\,\rho)\, d\omega\, \rho^2\, d\rho;$$

hence

$$\frac{\partial W}{\partial t} = ct^2 \Delta_x \iint\limits_{|\omega|=1} f(x+ct\omega)\, d\omega,$$

and

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x \left(c^2 t \int\limits_{|\omega|=1} f(x+ct\omega)\, d\omega \right) = c^2 \Delta u.$$

By (3.6), $u(x,0) = 0$, while

$$u_t(x,0) = \iint\limits_{|\omega|=1} f(x) d\omega = 4\pi f(x).$$

The function $v(x,t) = (tM_g)_t$ also satisfies the wave equation, (provided $g \in C^3(\mathbb{R}^3)$) since the wave equation has constant coefficients. It is clear that $v(x,0) = 0$; moreover,

$$v_t(x,t) = \frac{\partial^2}{\partial t^2} tM_g = c^2 \Delta t M_g = tc^2 \Delta M_g,$$

hence $v_t = 0$ at $t = 0$. We thus see that

$$u(x,t) = tM_{u_1} + (tM_{u_0})_t \tag{3.7}$$

is a solution of (3.1) in $\mathbb{R}^3$.

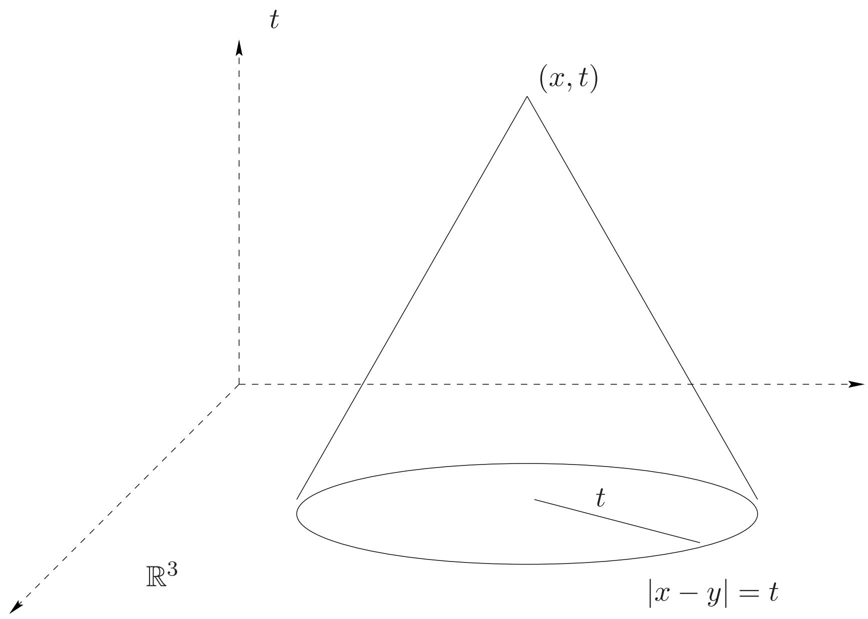


Figure 3.3: Backward light cone in $\mathbb{R}^3$

We leave it to the reader to verify that for $t > 0$, $tM_f = W_3 * f$, where

$$W_3(r,t) = \frac{\delta(r-ct)}{4\pi rc}; \tag{3.8}$$

hence W_3 is the fundamental solution of the wave equation in $\mathbb{R}^3$.

The fundamental property of the wave equation in three space dimensions known as *Huyghen's principle* follows immediately from Kirchhoff's representation. The spherical mean $M_f(x,t)$ depends only on the values of f on the sphere of radius ct centered at x; while $(tM_g)_t$ depends only on the values the normal derivative of g on this same sphere. The domain of dependence of the point (x,t) is therefore the backward light cone $|x-y| = ct$. This means that an observer at the origin observes a signal at time t only when the source of the signal is precisely at a distance ct away.

The situation in $\mathbb{R}^2$ is quite different. The solution of the wave equation in two space dimensions was originally obtained by Poisson; but it can be obtained very quickly from the three dimensional solution by the so-called method of descent, due to Hadamard, as follows.

A solution of the wave equation in $\mathbb{R}^2$ is regarded as a special case of the solution in three dimensions for which the initial data is independent of one of the spatial variables, say $x_3 = z$. The integration is carried out in cylindrical coordinates:

$$y_1 = \rho\cos\theta, \qquad y_2 = \rho\sin\theta, \qquad y_3 = z = \sqrt{c^2t^2 - \rho^2}.$$

The element of surface area is

$$dS = \sqrt{1 + (\nabla z)^2}\,\rho\,d\rho\,d\theta = \sqrt{1 + z_\rho^2}\,\rho\,d\rho\,d\theta.$$

Since $z_\rho = -\rho/z$, we have

$$dS = \sqrt{1 + \frac{\rho^2}{z^2}}\rho\,d\rho\,d\theta = \frac{1}{z}\sqrt{z^2 + \rho^2}\rho\,d\rho\,d\theta = \frac{ct}{z}\rho\,d\rho\,d\theta.$$

The upper and lower hemispheres make equal contributions, so tM_f reduces to the integral

$$2\frac{1}{4\pi c^2 t}\int_0^{ct}\int_0^{2\pi}\frac{f(x_1 + \rho\cos\theta, x_2 + \rho\sin\theta)}{\sqrt{c^2t^2 - \rho^2}}ct\,\rho\,d\rho\,d\theta.$$

This integral is a convolution $W_2 * f$, where

$$W_2 = \begin{cases} \dfrac{1}{2\pi c\sqrt{c^2t^2 - r^2}}, & r^2 < c^2t^2; \\ 0 & r^2 > c^2t^2. \end{cases}$$

It can be simplified to

$$W_2 * f = \frac{1}{2\pi c}\int_0^{ct} F(x;r)\frac{r}{\sqrt{c^2t^2 - r^2}}dr,$$

where

$$F(x;r) = \int_o^{2\pi} f(x_1 + r\cos\theta, x_2 + r\sin\theta)d\theta.$$

Finally, we may write W_2 in the compact form

$$W_2 = \frac{1}{2\pi c\sqrt{\sigma_+}}, \qquad \sigma = c^2t^2 - r^2 \tag{3.9}$$

where σ_+ denotes the positive part of σ, defined to be σ when $\sigma > 0$ and 0 when $\sigma < 0$.

Let us verify that the convolution integral satisfies the initial conditions

$$\Big(W_2 * f\Big)_t(x,0) = f(x).$$

Under the change of variables $r = \rho ct$ the integral above becomes

$$\frac{t}{2\pi}\int_0^1 \frac{\rho}{\sqrt{1-\rho^2}} F(x;\rho ct)d\rho$$

$$= \frac{tF(x;0)}{2\pi}\int_0^1 \frac{\rho}{\sqrt{1-\rho^2}} d\rho + O(t^2) = tf(x) + O(t^2),$$

The derivative of this integral at $t = 0$ is therefore $f(x)$.

Note that the domain of dependence of the point (x,t) in 2 space dimensions is the *entire* region $|y - x| \leq ct$! Thus, in three dimensions, a flash of light, or a thunderclap immediately passes by the observer, while in two dimensions, the disturbance persists forever. For example, consider the result of dropping a stone into a calm (infinite, for the sake of this discussion) pool of water. The ripples persist forever, although they do decay.

3.2 Characteristic curves in the plane.

Consider the simple equation

$$u_t + u_x = 0, \tag{3.10}$$

whose general solution is $u(x,t) = f(x-t)$. Any solution of this partial differential equation is constant along the lines $x - t$ =constant. These lines are called the *characteristics* of the equation. We see that u cannot be specified arbitrarily along a characteristic.

Moreover, we can formally extend the notion of solution of (3.10) to include any function of $x - t$, even a discontinuous one. For example, we might consider the "solution" $u(x,t) = H(t - x)$, where H is the Heaviside step function. This solution has a jump discontinuity across the characteristic $x = t$.

Even though the function $H(t - x)$ is discontinuous, it satisfies (3.10) in the following *weak* sense. For any given domain Ω in the plane, multiply (3.10) by a smooth function φ with compact support in Ω, (for example, $\varphi \in C_0^1(\Omega)$), and integrate by parts. We get

$$\iint_\Omega u(\varphi_x + \varphi_t)\, dx\, dt = 0, \qquad \forall \varphi \in C_0^1(\Omega). \tag{3.11}$$

Conversely, if (3.11) holds on some domain Ω and $u \in C^1(\Omega)$ then, integrating by parts, we find that

$$\iint_\Omega (u_t + u_x)\varphi\, dx\, dt = 0 \qquad \forall \varphi \in C_0^1(\Omega).$$

It follows that $u_t + u_x$ must vanish everywhere in Ω, i.e. that (3.10) holds in Ω.

Thus we see that the characteristics for (3.10) have the following two properties:

- A solution cannot be specified arbitrarily along a characteristic;
- A solution may have jump discontinuities across a characteristic.

The notion of a jump discontinuity requires the concept of a weak solution of the partial differential equation.

Theorem 3.2.1 *Let $f(\xi)$ be a piecewise differentiable function with a jump discontinuity $[f]$ at $\xi = 0$, i.e.*

$$[f] = \lim_{\xi \to 0+} f(\xi) - \lim_{\xi \to 0-} f(\xi).$$

Then $u(x,t) = f(x-t)$ is a weak solution of (3.10)*; that is, it satisfies* (3.11)*.*

Proof: Let Ω be any domain in $t > 0$. It is clear that u satisfies (3.10) in the strong sense in either of the domains $x < t$ or $x > t$; hence (3.11) holds whenever Ω lies on one side of the line $\Gamma := \{x = t\}$. Therefore, let Ω be any domain which intersects the line $x = t$, and let $\Omega_1 = \Omega \cap \{x < t\}$ and $\Omega_2 = \Omega \cap \{x > t\}$. We apply Green's theorem to each of the domains Ω_1 and Ω_2 separately. (See Figure (3.2).) In each

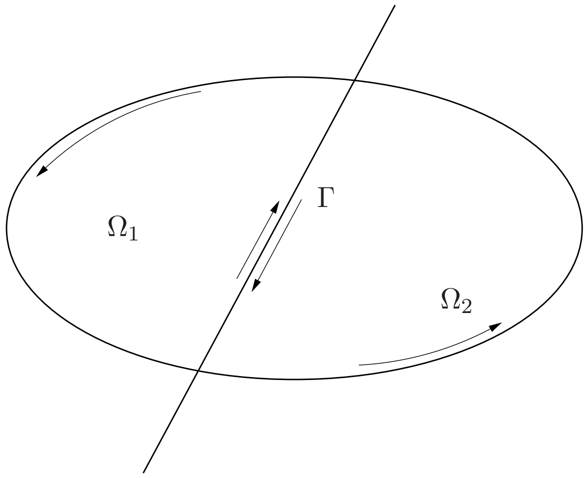


Figure 3.4: Jump conditions across a characteristic.

subdomain, u satisfies (3.10) in the strong sense. Since φ vanishes on $\partial\Omega$ we get

$$\iint_{\Omega} u(\varphi_x + \varphi_t)\,dxdt = \iint_{\Omega_1\cup\Omega_2} (u\varphi)_x + (u\varphi)_t\,dxdt = \int_{\Gamma} \varphi[f](dt - dx).$$

Since $dx/dt = 1$ on Γ, the one-form $dt - dx = 0$ on Γ. ■

A *non-characteristic* curve is any curve Γ which is never tangential to any characteristic; in this case, the slope of Γ is never equal to 1. If u satisfies (3.10) and is specified along a non-characteristic curve Γ, then u is determined uniquely everywhere in the plane.

More generally, we may consider the equation

$$u_t + c(x,t)u_x = 0, \tag{3.12}$$

in which we assume that c is a C^1 function. In this case the characteristics are given as solutions of the ordinary differential equation

$$\frac{dx}{dt} = c(x,t), \qquad x(s,0) = s.$$

Thus $x(s,t)$ is parametrized by the time variable t and the point at which the curve intersects the x axis. Along any such curve, u is constant, for

$$\frac{d}{dt}u(x(s,t),t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + cu_x = 0.$$

Now consider the wave equation in one dimension, (3.1). We have seen that its general solution is $u(x,t) = f(x-t)+g(x+t)$. 6mm 6mm

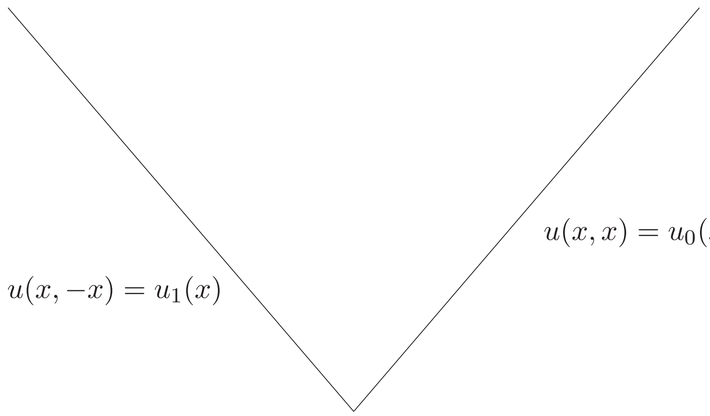


Figure 3.5: The Goursat problem

We want to solve the wave equation in the region $t^2 > x^2$ with data specified along the characteristics $x = \pm t$:

$$u(x,x) = u_0(x), \qquad u(x,-x) = u_1(x),$$

where u_0 and u_1 are given. We assume that u_0 and u_1 are C^2 and that $u_1(0) = u_0(0) = a_0$, so that the boundary data is continuous at the origin. Thus, we obtain

$$\begin{aligned} u(x,x) = f(0) + g(2x) =& u_0(x), \\ u(x,-x) = g(0) + f(2x) =& u_1(x). \end{aligned}$$

Setting $x = 0$ in these two equations, we see that we must have

$$f(0) + g(0) = u_0(0) = u_1(0) = u(0,0).$$

We then find

$$g(x) = u_0\left(\frac{x}{2}\right) - f(0), \qquad f(x) = u_1\left(\frac{x}{2}\right) - g(0),$$

hence

$$u(x,t) = u_1\left(\frac{x-t}{2}\right) + u_0\left(\frac{x+t}{2}\right) - u(0,0).$$

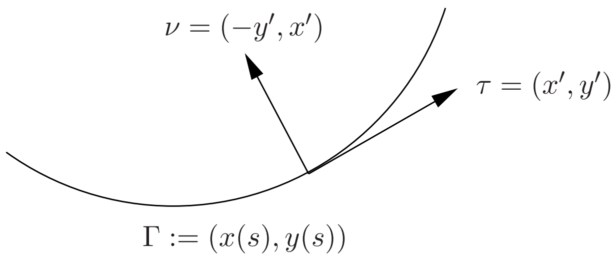


Figure 3.6: Conditions along a non-characteristic curve

This is called the characteristic, or *Goursat* problem.

We cannot, however, specify *both* u and u_t on a characteristic, for this would lead to an overdetermination of one of the functions f or g. For example, if we try to specify both u and u_t on the line $x = t$, we are led to

$$u(x,x) = f(0) + g(2x), \qquad u_t(x,x) = g'(2x) - f'(0);$$

and now both $g(2x)$ and $g'(2x)$ are determined. This is in general impossible.

With this example in mind, let us consider a general second order equation in two variables:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} = 0, \qquad (x,y) \in \Omega.$$

Let Γ be a C^2 curve lying in Ω, and suppose we try to specify both u and its normal derivative u_ν on Γ. Specifying u on Γ determines its tangential derivative u_τ along the curve. If Γ is parametrized by $x(s),\ y(s)$, where s is the arc length along the curve, then

$$u_\tau(s) = \frac{d}{ds}u(x(s), y(s)) = u_x x' + u_y y',$$

$$u_\nu(s) = \nabla u \cdot \nu = -u_x y' + u_y x',$$

where $x' = dx/ds$, etc, are both known functions on Γ. Since s is the arc length, the matrix

$$\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix}$$

has determinant 1 and is invertible; hence both u_x and u_y are uniquely determined along Γ by the values of u and u_ν along the curve.

We can therefore compute the tangential derivatives of u_x and u_y along Γ. These, along with the partial differential equation itself, give a set of equations for the second derivatives of u:

$$x'u_{xx} + y'u_{xy} = \frac{du_x}{d\tau}$$

$$x'u_{xy} + y'u_{yy} = \frac{du_y}{d\tau}$$

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0.$$

These equations for u_{xx}, u_{xy} and u_{yy} are solvable provided the determinant of the 3×3 matrix of coefficients does not vanish. In that case the curve Γ is called non-characteristic, and u and a non-tangential derivative can be prescribed independently along the curve.

The characteristics are determined by the vanishing of the determinant of the 3×3 system above, that is, by the condition

$$Ay'^2 - 2Bx'y' + Cx'^2 = 0.$$

This equation has two real solutions iff $B^2 - AC > 0$. Using

$$\frac{dy}{dx} = \frac{y'}{x'}$$

we can write the equation as

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0, \qquad \frac{dy}{dx} = B \pm \sqrt{B^2 - AC}. \tag{3.13}$$

We thus obtain a pair of ordinary differential equations for the characteristics. For example, in the case of the standard wave equation, $B = 0$ and $AC = -1$; so the characteristics are given as solutions of the ordinary differential equations

$$\frac{dy}{dx} = \pm 1.$$

It is impossible to prescribe, independently, both the function and a non-tangential derivative along the curve, since these are in general incompatible with the differential equation.

Second order equations in two variables with two real characteristics are called *hyperbolic* equations. An equation is called *elliptic* if it has no real characteristics, and *parabolic* if it has precisely one real characteristic. Laplace's is elliptic, and the heat equation is parabolic.

3.3 Characteristic surfaces in higher dimensions.

The determination of characteristic surfaces in higher dimensions is somewhat more complicated, but proceeds from the same notion. Consider a general second order partial differential equation

$$L[u] = \sum_{j,k=1}^{n} g_{jk} u_{jk} + \sum_{j=1}^{n} b_j u_j + cu = 0 \tag{3.14}$$

where $u_j = \partial u / \partial x_j$, etc. We assume throughout the discussion that the coefficients g_{jk} ... are smooth functions of x in some domain $\Omega \in \mathbb{R}^n$.

A smooth hypersurface C of codimension 1 lying in Ω is said to be non-characteristic for L if for any solution u defined in some neighborhood of C, both u and its normal derivative u_ν can be specified independently on C. Otherwise C is said to be a *characteristic surface* for the operator L.

We assume C is given as the level set $\varphi(x_1, \dots, x_n) = 0$ of some differentiable function φ, with $\nabla\varphi \neq 0$ in a neighborhood of C. Given u on C we can compute its $n-1$ independent tangential derivatives, denoted by $\tau_1 u, \dots \tau_{n-1} u$, on C. These, together with u_ν, uniquely determine all n first order derivatives $u_1, \dots, u_n$ on C.

Let us first consider the case in which $\varphi = x_n$ and ask when $x_n = 0$ is a characteristic surface. Write the equation as

$$L[u] = g_{nn} u_{nn} + \sum_{j=1}^{n} \sum_{k=1}^{n-1} g_{jk} u_{jk} + \sum_{j=1}^{n} b_j u_j + cu.$$

Since u and u_n are given on $x_n = 0$, the first derivatives u_j and the second order derivatives u_{jk}, $j = 1, \dots, n-1$, $k = 1, \dots, n$ are determined. Hence all terms in $L[u]$ are determined except $g_{nn} u_{nn}$.

If $g_{nn} = 0$ on $x_n = 0$, then (3.14) puts an additional constraint on u and u_n; i.e. the system is overdetermined, and u, u_n cannot be specified independently. On the other hand, if $g_{nn} \neq 0$ on $x_n = 0$, then u, u_n can be specified independently, and u_{nn} can be determined from (3.14).

Thus, the hypersurface $x_n = 0$ is characteristic if and only if $g_{nn} \equiv 0$ there.

We now turn to the general case. Let the surface C be given by $\varphi = 0$ and make a smooth, invertible coordinate transformation

$$y_n = \varphi(x_1, \dots x_n); \qquad y_j = y_j(x_1, \dots, x_n), \qquad j = 1, \dots, n-1.$$

The partial differential equation transforms as follows. Define w by $w(y) = u(x)$. Then

$$u_j = w_l \frac{\partial y_l}{\partial x_j} \qquad u_{jk} = w_{lm} \frac{\partial y_l}{\partial x_j} \frac{\partial y_m}{\partial x_k} + w_l \frac{\partial^2 y_l}{\partial x_j \partial x_k},$$

and

$$\sum_{j,k=1}^{n} g_{jk} u_{jk} = \sum_{l,m=1}^{n} \widetilde{g}_{lm} w_{lm} + \dots$$

where $\dots$ denotes terms depending on $w_1, \dots w_n$, and

$$\widetilde{g}_{lm} = \sum_{j,k=1}^{n} g_{jk} \frac{\partial y_l}{\partial x_j} \frac{\partial y_m}{\partial x_k}.$$

Since the coordinate transformation $x \mapsto y = y(x)$ is invertible, the w_j may be computed as functions of the u_j. Thus, the condition that $\varphi = 0$ be a characteristic surface is transformed into the equation $\widetilde{g}_{nn} = 0$, i.e.

Theorem 3.3.1 *The necessary and sufficient condition that the surface $\varphi = 0$ be a characteristic surface for the second order partial differential equation* (3.14) *is*

$$\sum_{j,k=1}^{n} g_{jk} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_k} = 0. \tag{3.15}$$

If $\varphi = 0$ is non-characteristic, then the second normal derivative $u_{\nu\nu}$ can be determined from the data u, u_ν on C and the equation $L[u] = 0$ in a neighorhood of C.

In the case of Laplace's equation, (3.15) becomes

$$(\nabla \varphi)^2 = 0,$$

which has no real non-trivial solutions. Laplace's equation is an example of an *elliptic* partial differential equation. It has no real characteristic surfaces.

However, (3.15) for the wave equation, with variables x, y, z, t, is

$$\varphi_t^2 - c^2 (\nabla \varphi)^2 = 0,$$

and this equation has the real non-trivial solution

$$\sigma = c^2t^2 - r^2, \qquad r^2 = x^2 + y^2 + z^2. \tag{3.16}$$

The level surfaces of σ are cones in four-dimensional space-time, called the light cones. These are the characteristic surfaces for the wave equation.

3.4 The wave equation in $\mathbb{R}^n$.

We saw in the first two chapters that the heat and Laplace equations have a smoothing property, in that solutions in the interior of the domain are highly differentiable even though the boundary data may not be differentiable. The situation is quite different for the wave equation, in that the solutions of the wave equation *lose* regularity. Even if the initial data for the wave equation is C^2, the solution may not be differentiable for $t > 0$. This loss of regularity increases with dimension, and one is forced to consider solutions in the class of distributions, even for smooth initial data. The solution of the wave equation in higher space dimensions leads to a class of divergent integrals whose correct interpretation, by HADAMARD [11], was one of the factors leading to the development of distribution theory[1].

We have seen that the fundamental solution of Laplace's equation in n dimensions is a scalar multiple of r^{-n}. The quantity r is invariant under the group of rotations about the origin, which also leaves the Laplacian invariant. The wave equation is invariant under the *Lorentz transformations*, the group of all linear transformations of the variables x, y, z, t which leave the quadratic form (3.16) invariant. We therefore look for a solution in the form $W = \Theta(\sigma)$.

We have

$$\Box W = \frac{\partial}{\partial t}\big(2c^2t\Theta'\big) - c^2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\big(-2x_j\Theta'\big) = 4c^2\Big(\sigma\Theta'' + \frac{n+1}{2}\,\Theta'\Big).$$

We seek the fundamental solution of the wave equation as a solution of the differential equation

$$\sigma\Theta'' + \frac{n+1}{2}\,\Theta' = 0 \tag{3.17}$$

[1]The results were originally presented in series of lectures at Yale University in 1921, under the auspices of the Silliman Foundation.

By direct integration, we obtain the solutions

$$\Theta_n = A_n \sigma^{\frac{1-n}{2}}.$$

When $n = 2$, this is the correct choice; however, when $n = 3$ we have already seen that the fundamental solution is a delta function, so we must be prepared to consider distribution solutions of (3.17). This is the pattern for even and odd n. We shall see that there is a simple recursion relation for obtaining the fundamental solutions in higher dimensions.

A function of $x \in \mathbb{R}^n$ is said to be homogeneous of degree α if $f(\lambda x) = \lambda^\alpha f(x)$. Differentiating this identity with respect to λ and setting $\lambda = 1$, we obtain Euler's equation

$$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = \alpha f.$$

Therefore (3.17) is satisfied by functions Θ for which Θ' is homogeneous of degree $-(n+1)/2$, and hence Θ_n is homogeneous of degree $(1-n)/2$. By differentiating (3.17) with respect to σ we obtain the equation for Θ_{n+2}. Thus, we may consider a recursion argument of the form $\Theta_{n+2} = C_n D\Theta_n$ for some constant C_n, where $D = d/d\sigma$.

The fundamental solution of the wave equation in $\mathbb{R}^1$, given by (3.4), is homogeneous of degree 0, as we should expect. The following theorem can be proved by the same argument used in the proof of theorem 1.2.1.

Theorem 3.4.1 *The fundamental solution of the wave equation in n space dimensions is homogeneous of degree $1 - n$ in $x_1, \ldots, x_n, t$.*

According to the proposed recursion argument, we should expect $W_3(\sigma)$ to be a multiple of $W_1'(\sigma)$. By (3.28) in the exercises below, we have, for $t > 0$, (δ is even)

$$\delta(\sigma) = \delta(r^2 - c^2t^2) = \frac{\delta(r - ct)}{2r}; \tag{3.18}$$

hence, by (3.8), $W_3(\sigma) = \pi^{-1} W_1'(\sigma)$. The fundamental solution for the wave equation in all odd dimensions n is obtained by subsequent differentiation, and in fact, we shall see that

$$W_{2m+1}(\sigma) = \left(\frac{1}{\pi}\frac{d}{d\sigma}\right)^m W_1 = \frac{1}{2\pi^m c}\delta^{[m-1]}(\sigma), \quad m = 0, 1, \ldots. \tag{3.19}$$

We shall leave it as exercises to show that

$$W_{2m+1} * f = \frac{1}{(2\pi)^m 2c^2 t} \left[\left(\frac{d}{dr} \frac{1}{r} \right)^{m-1} r^{2m} F(x;r) \right]_{r=ct} \tag{3.20}$$

$$F(x;r) = \iint\limits_{|\omega|=1} f(x+\omega r) d\omega; \tag{3.21}$$

and that

$$\frac{\partial}{\partial t}(W_{2m+1} * f)\Big|_{t=0} = f. \tag{3.22}$$

The solution of the wave equation in $n = 2m+1$ space dimensions thus involves derivatives of the initial data at the backward light cone up to order $m-1$.

The solution of the wave equation in even space dimensions raises some new issues. We may formally raise the dimension by one by taking a derivative of order 1/2. The fractional derivative of order $\alpha,\ 0 < \alpha < 1$, is defined by[2]

$$D^\alpha f = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-y)^{-\alpha} f(y)\, dy.$$

Observe that $D^{1/2}$ reduces the homogeneity of a function by 1/2. Let us assume that the fractional derivative can be extended to distributions, and that Θ_n is a distribution homogeneous of degree $(1-n)/2$. Then $D^{1/2}\Theta_n$ is homogeneous of degree

$$\frac{1-n}{2} - 1 = \frac{1-(n+1)}{2}.$$

Thus $D^{1/2}\Theta_n$ is a candidate, up to a scalar multiple, for Θ_{n+1}. In fact, we may obtain the fundamental solution of the wave equation in

[2]The justification for this nomenclature is obtained by taking the Laplace transform of this expression. Recall that if

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt$$

is the Laplace transform of f then $\mathcal{L}(Df) = s\mathcal{L}(f)$, provided that $f(0) = 0$. Using the convolution theorem for the Laplace transform, one finds that $\mathcal{L}(D^\alpha f)(s) = s^\alpha \mathcal{L}(f)(s)$.

$\mathbb{R}^2$ by computing $D^{1/2}H(\sigma)$. First note that $D^{1/2}H(\sigma)$ has support on $\sigma > 0$ since $H(\sigma)$ does. Therefore

$$\frac{1}{\Gamma(1/2)}\frac{d}{d\sigma}\int_0^\sigma \frac{H(y)}{2c}dy = \frac{1}{2\sqrt{\pi}c}\frac{d}{d\sigma}\int_0^\sigma \frac{dy}{\sqrt{\sigma - y}} = \frac{1}{2\sqrt{\pi}c}\frac{1}{\sqrt{\sigma_+}},$$

where σ_+ is the positive part of σ. Hence $W_2(\sigma) = \pi^{-1/2}W_1'(\sigma)$. In view of (3.19) we might expect that

$$W_{2m} = \left(\frac{1}{\pi}\frac{d}{d\sigma}\right)^{m-1} W_2 = \left(\frac{1}{\pi}\frac{d}{d\sigma}\right)^{m-1/2} W_1.$$

This is in fact true; and moreover, the results in even and odd dimensions can be combined into a single formula.

Theorem 3.4.2 *The fundamental solution for the wave equation in n dimensions is given by*

$$W_n = \left(\frac{1}{\pi}\frac{d}{d\sigma}\right)^{\frac{n-1}{2}} W_1. \tag{3.23}$$

We have already shown that the distribution in (3.23) is a solution of the wave equation in $\mathbb{R}^n$; so all that is needed is to verify that this distribution gives the correct solution to (3.1). This task has been left as an exercise.

For $n = 4$ we obtain $W_4 = -\sigma_+^{-3/2}/2\pi$, which leads to the divergent integral

$$\int_0^{ct} \iiint\limits_{|\omega|=1} \frac{f(x+\omega r)}{(c^2t^2 - r^2)^{3/2}} r^3 dr\, d\omega. \tag{3.24}$$

A good deal of effort was expended in the latter part of the 19^{th} century in trying to regularize such singular integrals. The ultimate resolution of the issue was obtained by Hadamard. He pointed out that in some cases such integrals can be regularized by various devices, but in the end he comments that *they would not be of interest to us, as – paradoxical as it may seem – our proposed method will consist in not avoiding them.*

Hadamard thus recognized that the integrals obtained are not to be regularized but must instead be reinterpreted. Classical mathematics had broken down for the wave equation in higher dimensions, and the divergent integrals signified that something new was afoot; in fact, Hadamard's bold solution of the problem was a major impetus in the development of the modern theory of distributions. The wave equation does not have classical solutions unless the initial data is sufficiently regular. Rather, it has the property that its solutions lose regularity, the loss of derivatives increasing with dimension. The solutions obtained are distributions rather than classical functions, for which the divergent integrals are symbols.

Hadamard introduced what he called the "finite part" of an integral such as (3.24). To illustrate the idea with a simple example, consider the divergent integral

$$\int_0^\infty y^{-3/2} f(y)\,dy$$

for a smooth function f with compact support on the positive real line. We may try to interpret this integral by replacing the lower limit by $\varepsilon > 0$ and letting $\varepsilon \to 0+$. We have

$$\begin{aligned}\int_\varepsilon^\infty y^{-3/2} f(y)\,dy &= \int_\varepsilon^\infty y^{-3/2}(f(y)-f(0))\,dy + f(0)\int_\varepsilon^\infty y^{-3/2}\,dy \\ &= \int_\varepsilon^\infty y^{-3/2}(f(y)-f(0))\,dy + 2\frac{f(0)}{\sqrt{\varepsilon}}.\end{aligned}$$

If $|f(y)-f(0)| \leq K|y|^\alpha$, with $\alpha > 1/2$, then the first term has a well-defined limit as $\varepsilon \to 0+$. The second term, however, is unbounded as $\varepsilon \to 0$ whenever $f(0) \neq 0$, and Hadamard simply threw this term away, defining the integral to be given by its *finite part*[3]:

$$p.f. \int_0^\infty y^{-3/2} f(y)\,dy = \int_0^\infty y^{-3/2}(f(y)-f(0))\,dy.$$

Hadamard constructed an elaborate calculus of such divergent integrals, extending the theory to multiple dimensions, and applying it to the wave equation.

In the modern theory of distributions, Hadamard's device is replaced by writing $x^{-3/2}$ as the derivative of $-2x^{-1/2}$, and integrating

[3]Hadamard's lectures were written in English, quite good English, as a matter of fact; they were later translated into French by Mlle J. Hadamard.

the improper integral by parts to obtain

$$p.f. \int_0^\infty y^{-3/2} f(y)\, dy = \int_0^\infty y^{-3/2}(f(y) - f(0))\, dy$$

$$= -2\Big(y^{-1/2}(f(y) - f(0))\Big)\Big|_0^\infty + 2\int_0^\infty \frac{1}{\sqrt{y}} f'(y)\, dy$$

$$=2\int_0^\infty \frac{1}{\sqrt{y}} f'(y)\, dy.$$

Let us return to our discussion of the wave equation in even space dimensions. The solution of the wave equation in $2m$ dimensions is formally represented by the divergent integral

$$\int_0^{ct} \left[\left(\frac{1}{\pi}\frac{d}{d\sigma}\right)^{m-1} W_2\right] F(x;r) r^{2m-1} dr.$$

As we saw above, this integral is interpreted in the sense of distributions by formally integrating by parts and ignoring infinite terms.

For constant t,

$$\frac{d}{d\sigma} = -\frac{1}{2r}\frac{d}{dr}, \tag{3.25}$$

hence the integral above is interpreted as

$$\frac{1}{(2\pi)^m c}\int_0^{ct}\left[\left(-\frac{1}{r}\frac{d}{dr}\right)^{m-1}\frac{1}{\sigma_+}\right] F(x;r) r^{2m-1} dr$$

$$=\frac{1}{(2\pi)^m c}\int_0^{ct}\frac{1}{\sigma_+}\left[\left(\frac{1}{r}\frac{d}{dr}\right)^{m-1} F(x;r) r^{2m-1}\right] dr. \tag{3.26}$$

For initial data of class C^{m-1} the expression (3.26) is an ordinary improper integral.

We leave it as an exercise to verify that (3.26) satisfies the appropriate initial conditions. The formula for the surface area of the unit sphere in $\mathbb{R}^n$, denoted by ω_n, will be needed; it is given by

$$\omega_n = A(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{3.27}$$

3.5 Exercises

1. Let $x_0 = ct, x_1 = x, x_2 = y, x_3 = z$; we write $x_\mu = (x_0, x_1, x_2, x_3)$, and x is called a 4-vector. Find a matrix g such that $\sigma = g^{\mu\nu} x_\mu x_\nu$. Define an inner product of 4-vectors by $x \cdot y = g^{\mu\nu} x_\mu y_\nu$. Denote Lorentz transformations by $x' = \Lambda x$, where x and x' are 4-vectors. The Lorentz group is the group of all 4×4 matrices Λ which preserve the inner product, i.e. $x \cdot y = x' \cdot y'$. Let the D'Alembertians in the two coordinate systems be denoted by $\Box$ and $\Box'$. Prove that $\Box = \Box'$, hence the wave equation is invariant under Lorentz transformations.

2. Prove the following statements:

   (a) the delta function on the line is homogeneous of degree -1;

   (b) the delta function is homogeneous of degree $-n$ in $\mathbb{R}^n$;

   (c) the derivative of a homogeneous function of degree α is homogeneous of degree $\alpha - 1$;

   (d)
   $$\int_0^\infty f(x)\delta(x)dx = \frac{1}{2}f(0).$$

   Find the delta function in spherical coordinates in $\mathbb{R}^n$. Hint: Represent the delta function as the limit

   $$\delta(x) = \lim_{\varepsilon \downarrow 0} \frac{e^{-x^2/4\varepsilon}}{\sqrt{4\pi\varepsilon}}.$$

3. Let $f(x)$ be a smooth function such that $f(x_0) = 0,\ f'(x_0) \neq 0$. Show that in the vicinity of x_0,

   $$\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|. \tag{3.28}$$

   Use this result to establish (3.5) and (3.8)

4. Derive (3.20) using (3.25). Verify that the solution v given by integrals (3.20) and (3.26) satisfies the initial conditions $v(x, 0) = 0,\ v_t(x, 0) = f(x)$.

Solution: For $n = 2m+1$, first prove that

$$\left(\frac{d}{dr}\frac{1}{r}\right)^{m-1} r^{2m} =(2m-1)\cdots 3r^2 = \frac{2^m\Gamma(m+1/2)}{\sqrt{\pi}}r^2$$

$$=\frac{(2\pi)^m\Gamma(m+1/2)}{\pi^{m+1/2}}r^2 = \frac{2(2\pi)^m}{\omega_n}r^2.$$

Then

$$W_{2m+1} * f =\frac{1}{(2\pi)^m 2c^2 t}\left(\frac{d}{dr}\frac{1}{r}\right)^{m-1} r^{2m}F(x;r)\Big|_{r=ct}$$

$$=\frac{t}{\omega_n}F(x;ct) + O(t^2),$$

and the result follows. For $n = 2m$ we have

$$\left(\frac{d}{dr}\frac{1}{r}\right)^{m-1} r^{2m-1} = 2^{m-1}(m-1)!r = 2^{m-1}\Gamma(m)r,$$

hence by (3.27)

$$\frac{1}{(2\pi)^m c}2^{m-1}\Gamma(m)\int_0^{ct}\frac{r}{\sqrt{c^2t^2-r^2}}F(x;r)dr = \frac{t}{\omega_{2m}}F(x;ct)+O(t^2).$$

5. Given the boundary-initial value problem

$$u_{tt} - \Delta u + h(u) = 0, \quad x \in \Omega,$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad u(\cdot,t)\Big|_{x\in\partial\Omega} = 0,$$

prove that the energy

$$\mathcal{E} = \iiint\limits_{\Omega} \frac{1}{2}(u_t^2 + \nabla u^2) + H(u)\,dx, \qquad H(u) = \int\limits_0^u h(s)\,ds$$

is conserved.

6. Let u satisfy the wave equation in $\mathbb{R}^3$ with $c = 1$, and let

$$\mathcal{E}(0) = \frac{1}{2} \iiint\limits_{|x-y|=\leq R} u_t^2 + (\nabla u)^2 \, dx \Big|_{t=0};$$

$$\mathcal{E}(t) = \frac{1}{2} \iiint\limits_{|x-y|+t=\leq R} u_t^2 + (\nabla u)^2 \, dx,$$

Prove that $\mathcal{E}(t) \leq \mathcal{E}_0$. Use this to prove the following uniqueness theorem for the wave equation: If u is a C^2 solution of the wave equation in $\mathbb{R}^3$ which vanishes in the ball $|x| \leq R$ at time $t = 0$ then u vanishes identically in the solid cone $|x| + t \leq R$.

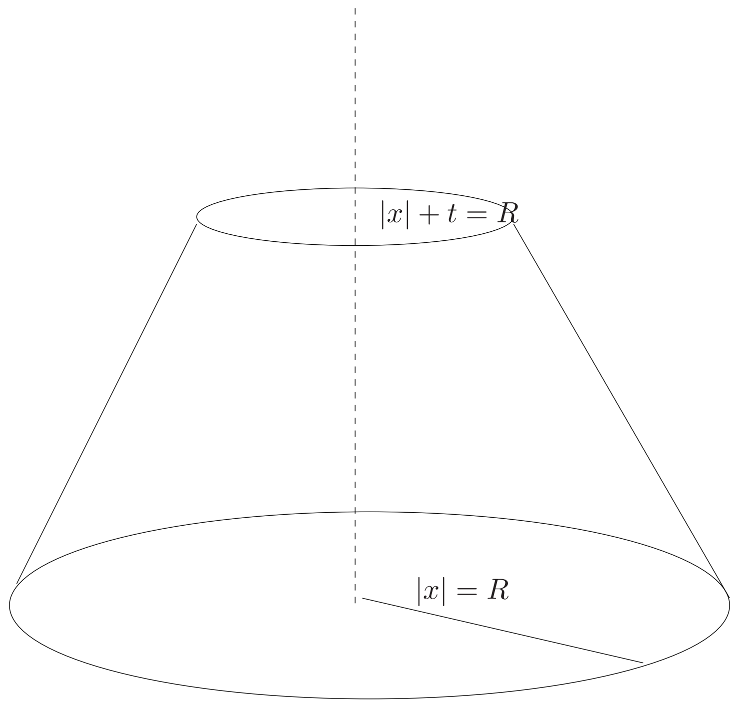


Figure 3.7: Backward ray cone in three dimensions

7. Assuming C^∞ initial data, solve the equation

$$u_t + uu_x = 0$$

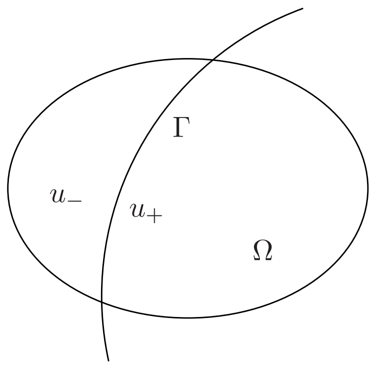


Figure 3.8: Shock discontinuity of a nonlinear hyperbolic equation

by the method of characteristics. Show that, unless the initial data is monotone, discontinuities in the solution must develop. Write the equation in weak form. Suppose that a weak solution exists which is discontinuous along a smooth curve Γ in the x-t plane given by $x = x(t)$, as in Figure (7). Suppose the limits of u on either side of Γ are denoted by u_+ and u_-. Show that the speed of the 'shock' Γ is given by

$$\dot{x} = \frac{u_+ + u_-}{2}.$$

Chapter 4

Equations of Fluid and Gas Dynamics

4.1 Conservation Laws

We derive in this section the equations of fluid mechanics and a few of the associated conservation laws. We focus largely on the theory of inviscid, irrotational flows, which forms the core of the classical theory of fluid mechanics. The modern theory of fluid mechanics deals with the effects of viscosity and the resulting boundary layers and turbulence in the vicinity of the boundaries of the flow domain. For a more detailed account of the subject, the reader may consult a number of texts and monographs, e.g. Batchelor [5], and Landau and Lifshitz, [12]; the treatise by Serrin [21] offers an historical and mathematical perspective of the subject.

The equations of gas dynamics are the expression of conservation of mass, momentum, and energy in differential form.

Let ρ and u (u is a vector field) denote the mass density and velocity of a compressible fluid. We assume these functions to be C^1 and we take Ω to be a domain with smooth boundary in $\mathbb{R}^3$. The particles follow trajectories given by the ordinary differential equations

$$\frac{dx^i}{dt} = u_i(x, t).$$

The particles may be labelled by their positions at some reference time, e.g. $t = 0$, by ξ and their positions at time t by $x = x(t, \xi)$. The variables ξ are called the material coordinates and the variables x are called the spatial coordinates. The trajectories of the particles are called the streamlines. The acceleration of the particles is given by the second derivative:

$$\frac{d^2x^i}{dt^2} = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x^j}\frac{dx^j}{dt} = \frac{\partial u_i}{\partial t} + u^j\frac{\partial u_i}{\partial x^j}.$$

If $F(x,t)$ is a scalar valued function, its *total* or *material derivative*, moving with the fluid flow, is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{j=1}^{3}\frac{\partial F}{\partial x^j}\dot{x}^j = \frac{\partial F}{\partial t} + \sum_{j=1}^{3} u^j\frac{\partial F}{\partial x^j}.$$

The equation

$$\frac{dF}{dt} = 0$$

means that F is constant along streamlines. The conservation laws of fluid mechanics take precisely this form.

The conservation laws are expressed in terms of rates of change of the mass, momentum, and energy contained in a given region of the flow domain. For example, the total mass contained in Ω is

$$\iiint\limits_{\Omega} \rho(x,t)\,dv,$$

where dv denotes the volume element $dx^1dx^2dx^3$. If Ω moves with the flow there is no mass flux across the boundary, and conservation of mass requires that

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho\,dv = 0.$$

Theorem 4.1.1 *When $\Omega = \Omega(t)$ moves with the flow given by $\dot{x}^i = u_i(x,t)$ we have*

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho\,dv = \iiint\limits_{\Omega}\left(\frac{d\rho}{dt} + \rho\operatorname{div} u\right)dv. \tag{4.1}$$

Proof: The particle trajectories are given by $x^i(t) = x^i(t,\xi)$, where $x^i(t,0) = \xi^i$. Write the integral in terms of the material variables ξ. Then we must calculate

$$\frac{d}{dt}\iiint\limits_{\Omega_0} \rho J\, d\xi, \qquad J = \frac{\partial(x^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)}$$

where Ω_0 is a fixed domain parameterized by the ξ variables. We say that the integral has been "pulled back" to the base manifold Ω_0. The theorem follows from the fact that

$$\dot{J} = J\mathrm{div}u. \tag{4.2}$$

and then transforming back to the coordinates x.

Equation (4.2) is proved as follows.

$$\begin{aligned}\dot{J} =& \frac{\partial(\dot{x}^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} + \frac{\partial(x^1,\dot{x}^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} + \frac{\partial(x^1,x^2,\dot{x}^3)}{\partial(\xi^1,\xi^2,\xi^3)}\\ =& \frac{\partial(u^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} + \frac{\partial(x^1,u^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} + \frac{\partial(x^1,x^2,u^3)}{\partial(\xi^1,\xi^2,\xi^3)}.\end{aligned}$$

By the chain rule,

$$\frac{\partial(u^1,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} = \sum_{j=1}^{3} \frac{\partial u^1}{\partial x^j}\frac{\partial(x^j,x^2,x^3)}{\partial(\xi^1,\xi^2,\xi^3)} = \frac{\partial u^1}{\partial x^1}J,$$

etc. The other two terms are computed in the same way. This completes the proof of (4.2) and hence the theorem. ■

The equation governing the conservation of mass follows immediately. Since Ω is an arbitrary domain, we must have

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,u = \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\, u) = 0. \tag{4.3}$$

Theorem 4.1.1 and its proof (due to Euler [21]) are a template for a much more general result, known as the *transport theorem*, which is fundamental to the derivation of conservation laws. The transport theorem provides a means to calculate the rate of change of integrals over curves or surfaces moving with the flow. Since differential forms are the natural objects for integration theory in higher dimensions, we need to calculate the action of the total derivative on differential forms.[1]

A $p-$form is integrated over a $p-$dimensional manifold. For example, the line integral of the 1-form $f_i dx^i$ over a path γ gives the work done by a force field with components f_i. The integral of the 3-form $\rho\, dx^1 \wedge dx^2 \wedge dx^3$ over a region Ω gives the total mass in Ω, and the integral of

$$\sum_{i<j} \left(\frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} \right) dx^i \wedge dx^j$$

over a surface S gives the vorticity flux through S.

These expressions, called *exterior differential forms*, have a number of advantages over their vectorial cousins. The operations curl, gradient, and divergence are not invariant under coordinate transformations, and moreover they are fundamentally tied to the metric tensor of the manifold, whereas the corresponding operation for differential forms, known as the *exterior derivative*, is defined in arbitrary dimensions, and takes the same form in any coordinate system. Moreover, the calculus of differential forms keeps track of the orientation of the manifolds of integration.

In the language of tensors, p-forms are covariant antisymmetric tensors of order p on a manifold M; they are denoted by $\Lambda(M)$. The basic rules of calculation of differential forms are extremely simple, and we summarize them here. We restrict our discussion to three operations: the wedge product, the exterior derivative, and the Lie derivative.

The wedge product is multilinear over scalar functions, anti-symmetric, and associative. Thus $(dx^1 \wedge dx^2) \wedge dx^3 = dx^1 \wedge (dx^2 \wedge dx^3)$; while

$$dx^1 \wedge dx^2 = -dx^2 \wedge dx^1, \qquad dx^1 \wedge f dx^2 = f dx^1 \wedge dx^2,$$

etc. In particular, $dx^1 \wedge dx^1 = 0$. The wedge

The *exterior derivative* d maps Λ_p to Λ_{p+1}. Its action on 0-forms (functions), is given by

$$df = \frac{\partial f}{\partial x^i}\, dx^i.$$

[1]The total derivative of a differential form is known in geometry as the Lie derivative with respect to the vector field u. For a more complete account, see Spivak [24] or Sattinger & Weaver [19].

Its action is uniquely extended to differential forms of any order by the three rules

- $d^2 = 0$;
- d is a linear map from Λ_p to Λ_{p+1};
- $d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^p \omega \wedge d\nu$ for $\omega \in \Lambda_p$.

The theorems of Green, Gauss, and Stokes are subsumed under a general theorem, known as Stokes' theorem

$$\iint\limits_{\Omega} d\omega = \int\limits_{\partial\Omega} \omega. \tag{4.4}$$

Here, ω is a p form with differentiable coefficients, and Ω is a $p+1$ dimensional manifold in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. It is possible to relax these regularity conditions somewhat, but we shall not need that for the present discussion.

We have already defined the total derivative relative to a vector field $\vec{u}$ on 0-forms, i.e. functions. The total derivative is extended to all differential forms by two simple rules:

- The Leibnitz rule:
$$\frac{d}{dt}\omega \wedge \nu = \frac{d\omega}{dt} \wedge \nu + \omega \wedge \frac{d\nu}{dt}.$$
- The total derivative and the exterior derivative commute:
$$\frac{d}{dt} d\omega = d\left(\frac{d\omega}{dt}\right)$$

Note that the total derivative of a $p-$form is again a $p-$form.

For example, the action of the total derivative on a one form $f_i dx^i$ is

$$\begin{aligned}\frac{d}{dt} f_i dx^i =& \frac{df_i}{dt} dx^i + f_i \frac{d}{dt} dx^i = \frac{df_i}{dt} dx^i + f_i d\dot{x}^i \\ =& \frac{df_i}{dt} dx^i + f_i du_i = \frac{df_i}{dt} dx^i + f_i \frac{\partial u_i}{\partial x^j} dx^j \\ =& \left(\frac{df_i}{dt} + f_j \frac{\partial u^j}{\partial x^i}\right) dx^i\end{aligned}$$

The total derivative of the mass density 3-form is

$$\frac{d}{dt}\rho\,dv = \frac{d\rho}{dt}dv + \rho(d\dot{x}^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge d\dot{x}^2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge d\dot{x}^3).$$

Now

$$\begin{aligned} d\dot{x}^1 \wedge dx^2 \wedge dx^3 =& du^1 \wedge dx^2 \wedge dx^3 \\ =& \left(\frac{\partial u^1}{\partial x^1}dx^1 + \frac{\partial u^1}{\partial x^2}dx^2 + \frac{\partial u^3}{\partial x^3}dx^3\right) \wedge dx^2 \wedge dx^3 \\ =& \frac{\partial u^1}{\partial x^1}dv \end{aligned}$$

etc., so

$$\frac{d}{dt}(\rho\,dv) = \left(\frac{d\rho}{dt} + \rho\,\mathrm{div}\,u\right)dv.$$

The transport theorem states:

Theorem 4.1.2 *Let ω be a $p-$form and $\Omega(t)$ a smooth $p-$dimensional manifold moving with the flow generated by the vector field u. Then*

$$\frac{d}{dt}\iint\limits_{\Omega} \omega = \iint\limits_{\Omega} \frac{d\omega}{dt}$$

The transport theorem is proved in the same way that Theorem 4.1.1 was proved: the integral is "pulled back" to the base manifold Ω_0, where $\Omega(t) = x(t, \Omega_0)$, and the differentiation carried out there.

Let us return to the derivation of the conservation laws. The conservation of linear momentum is derived by a similar argument. The momentum density in Ω is given by the vector field ρu, hence the rate of change of the i^{th} momentum component in the region Ω is[2]

$$\frac{d}{dt}\iiint\limits_{\Omega} \rho u_i\,dv = \iiint\limits_{\Omega} \frac{d}{dt}\Big(\rho\,u_i\,dv\Big).$$

Since the domain moves with the fluid, there is no transport of momentum across the boundary. By Newton's law of motion, the rate of

[2]This formula presupposes that momenta at different points in space can simply be added together, a fundamental precept of Euclidean geometry; this would not be the case in relativistic fluid mechanics, for example, if one were formulating the equations of fluid flow in the interior of a very dense object, such as a star

change of momentum is equal to the total force on the body. If we assume there are no internal stresses in the fluid (no viscosity), then the total force acting on the fluid particles is the sum of the hydrodynamic pressure acting on the boundary plus the integral of any external force (such as gravity) acting throughout the interior.

The total force on Ω in the i^{th} direction due to the hydrodynamic pressure is

$$-\iint\limits_{\partial\Omega} p\nu^i \, dS = -\iiint\limits_{\Omega} \frac{\partial p}{\partial x^i} \, dv;$$

and the i^{th} component of the total force on the fluid interior to Ω due to an external force f is

$$\iiint\limits_{\Omega} \rho f_i \, dv.$$

(The factor of ρ is required since f denotes the force per unit mass.)

Therefore, the conservation of the i^{th} component of the momentum for an inviscid fluid is

$$\iiint\limits_{\Omega} \frac{d}{dt}(\rho \, u_i \, dv) = \iiint\limits_{\Omega} \left(-\frac{\partial p}{\partial x^i} + \rho f_i \right) \, dv.$$

Since Ω is an arbitrary domain (with smooth boundary), we may conclude that

$$\frac{d}{dt}\rho \, u_i \, dv = u_i \frac{d}{dt}\rho \, dv + \rho \frac{du_i}{dt} dv = \left(-\frac{\partial p}{\partial x^i} + \rho f_i \right) \, dv.$$

We have already shown that the total derivative of $\rho \, dv$ vanishes, due to conservation of mass, hence this equation simplifies to[3]

$$\frac{du_i}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} = \frac{\partial u_i}{\partial t} + u^j \frac{\partial u_i}{\partial x^j} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} = f_i.$$

In terms of the total derivative, the equations of conservation of mass and momentum are therefore

$$\frac{d\rho}{dt} + \rho \, \mathrm{div} \, u = 0, \qquad \frac{du_i}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} = f_i. \tag{4.5}$$

[3]We use the summation convention here: repeated indices denote summation.

These equations are due to EULER in 1755[4]

The equations above are not closed, since we have only four equations in the five variables $\rho,\ u,\ p$. In order to close them we need either a fifth equation or a relationship between ρ and p. If the gas is locally in thermodynamic equilibrium, the pressure, temperature, and density are related by an equation of state $p = f(\rho, T)$. If the temperature can be assumed to be constant throughout the flow field, then the pressure and density are related by an equation $p = h(\rho)$. In this case the flow is said to be *barentropic*. For example, in the case of an ideal gas with constant specific heats, in which the entropy is constant (*isentropic flow*), the relationship between pressure and density is given by

$$p = N\rho^\gamma$$

for constants N and γ. For air, $\gamma = 7/5 = 1.4$

A second case of great importance in applications is that of incompressible flow. Water, for example, is essentially incompressible; but there are many applications, in geophysics for example, in which air flow can also be considered incompressible [5]. For incompressible flows, $\mathrm{div}\vec{u} = 0$ and the first of Euler's equations implies the density is constant. Euler's equations for an incompressible, inviscid fluid are

$$\frac{du_i}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} = 0, \qquad \frac{\partial u_i}{\partial x^i} = 0. \tag{4.6}$$

.

A one form $f = f_i dx^i$ is said to be *exact* if it is an exact differential of a function F, i.e. $f_i dx^i = dF$. In that case $f_i = \partial F/\partial x^i$, and F is said to be the potential. In vector analysis, a vector field $\vec{f}$ is said to be a conservative vector field if $\vec{f} = -\nabla F$. In the following we use one-forms rather than vector fields; then if the external force field is conservative, we have $f = -dF$ (the minus sign is in keeping with convention).

Similarly, a potential flow is one for which the velocity field is a gradient; i.e. $\vec{u} = \nabla\varphi$. We shall work with the corresponding one-forms and say that the flow is a potential flow if the one-form $u_i dx^i$ is exact, i.e. if $u_i dx^i = d\varphi$.[5] A necessary and sufficient condition that u

[4] L. EULER *Opera Omnia* II **12**. cf. SERRIN *op. cit.*

[5] We write the one-form as $u_i dx^i$ where u_i are the components of the covariant tensor obtained from the contravariant tensor u^i by lowering the indices; in Cartesian coordinates, however, $u_i = u^i$.

be a potential flow is that the path integral

$$\Gamma = \int\limits_C u_i\, dx^i,$$

called the *circulation*, vanish for any closed contour C. In that case we can construct the potential via the integral

$$\varphi(x) = \int\limits_{x_0}^{x} u_i dx^i.$$

Theorem 4.1.3 *The circulation around any closed curve C moving with the flow is conserved for barentropic flow with a conservative force field. Consequently, if the flow is initially a potential flow, then it remains a potential flow.*

Proof: By the transport theorem the rate of change of Γ along the flow is

$$\begin{aligned}\dot{\Gamma} =& \frac{d}{dt}\int\limits_C u_i\, dx^i = \int\limits_C \frac{du_i}{dt}\, dx^i + u_i \frac{d}{dt} dx^i \\ =& \int\limits_C \left(-\frac{1}{\rho}\frac{\partial p}{\partial x^i} + f_i\right) dx^i + u_i\, du_i \\ =& \int\limits_C -\frac{dp}{\rho} - dF + \frac{1}{2}d(u^2) = \int\limits_C -\frac{dp}{\rho},\end{aligned}$$

where $f_i = -\partial F/\partial x^i$. The last integral also vanishes, since we may write

$$\frac{dp}{\rho} = h'(\rho)d\rho = dh, \qquad h'(\rho) = \frac{p'(\rho)}{\rho}.$$

Therefore the circulation around any closed contour is invariant under the flow, and Γ is conserved. ■

By Stokes' theorem, $\Gamma = \iint\limits_{\Omega} \omega$ where ω is the two form

$$\omega = du = du_i \wedge dx^i = frac12 \sum_{i<j} \left(\frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j}\right) dx^i \wedge dx^j.$$

The 2-form ω is called the *vorticity*, though in classical fluid mechanics, the vorticity is taken to be the vector field $\omega = \nabla \times u$. The flow is said to be *irrotational* if the vorticity vanishes. By the calculation used in the proof of Theorem (4.1.3) one can show that

$$\frac{d\omega}{dt} = 0$$

when the external force field is conservative. Hence if the flow is initially irrotational it remains so.

If the circulation around any closed path vanishes, then the vorticity of the flow vanishes; but in a multiply connected domain, the vorticity may vanish everywhere yet the circulation is non-zero. We said above that a one form is exact if it is an exact differential. Thus the one form $u = u_i dx^i$ is exact if $u = d\varphi$; in that case the circulation always vanishes, and the flow is a potential flow.

On the other hand, a one form is *closed* if its exterior derivative vanishes. Thus u is closed if $\omega = du = 0$. In this case the flow is irrotational. In the language of differential forms, a flow is irrotational if $u_i dx^i$ is closed; and the flow is potential if $u_i dx^i$ is exact.

If the flow is irrotational then it can be written locally as the gradient of a potential; but unless the domain is simply connected, the potential may not be single valued. The best known example is the planar flow given by $\varphi = \theta$, where $\theta = \arctan y/x$ is the angular coordinate. The function θ is a local potential for the flow

$$u = d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2} = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy.$$

This flow is irrotational, since $du = d^2\theta = 0$; but the circulation around any closed curve containing the origin is

$$\Gamma = \int_C d\theta = 2\pi.$$

4.2 Bernoulli's Theorem

Daniel Bernoulli derived, in 1738, an energy conservation theorem in which the fluid velocity is treated as kinetic energy, while the hydrodynamic pressure is viewed as a potential energy. It sets forth the fundamental property of fluid flow in which variations in fluid velocity generate pressure differentials across a surface resulting in forces on airfoils, sails, etc.

Theorem 4.2.1 *Consider a steady incompressible flow in the presence of a conservative external force field with potential* $-F$*. Let* $u^2 = \sum_i u^{i,2}$*. Then*

$$\frac{1}{2}u^2 + \frac{p}{\rho} + F$$

is constant along streamlines.

Proof: Write the momentum equation in Lagrangian form

$$\ddot{x}^i + \frac{\partial}{\partial x^i}\left(\frac{p}{\rho} + F\right) = 0.$$

Multiplying these equations by $\dot{x}^i$ and summing, we obtain

$$\frac{d}{dt}\left(\sum_{i=1}^{3} \frac{x^{i,2}}{2} + \frac{p}{\rho} + F\right) = 0.$$

The result follows by replacing $\dot{x}^i$ by u_i. ■

A similar result holds for the flow of a compressible gas, namely

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} + F$$

is constant along streamlines. We leave this as an exercise.

Bernoulli's theorem holds for *any* steady flow, not just potential flows. In particular, it holds even in the presence of vorticity. A second conservation theorem, quite similar to Bernoulli's result, holds for unsteady potential flows.

Theorem 4.2.2 *Let* $\vec{u}(x,t)$ *be an incompressible potential flow in the presence of a conservative force field with potential* $-F$*. Let the velocity potential be denoted by* $\varphi(x,t)$*. Then, up to an additive factor* $C(t)$ *in the velocity potential, the equation*

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + \frac{p}{\rho} + F = const. \tag{4.7}$$

holds throughout the domain of flow.

Proof: Multiplying the momentum equation in (4.5) by dx^i and summing, we obtain

$$\left(\frac{du_i}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x^i} + \frac{\partial F}{\partial x^i}\right) dx^i = 0.$$

Now

$$\begin{aligned} \frac{du_i}{dt}dx^i =& \frac{d}{dt}u_i dx^i - u_i\frac{dx^i}{dt} = \frac{d}{dt}d\varphi - u_i du_i \\ =& d\left(\frac{d\varphi}{dt} - \frac{1}{2}u^2\right) = d\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2\right). \end{aligned}$$

Therefore we obtain

$$d\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} + F\right) = 0.$$

It follows that the quantity in parentheses on the left is spatially independent, hence

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} + F = C(t).$$

Replacing the velocity potential φ by

$$\varphi + \int_0^t C'(s)ds$$

we obtain (4.7). ■

Equation (4.7) plays a fundamental role in the dynamics of free surface problems, for example, in the analysis of wave motion on the surface of a body of water.

Using Bernoulli's theorem, let us compute the force on an object by a two dimensional incompressible flow whose asymptotic behavior as $r \to \infty$ is given by

$$\mathbf{u} \sim U\mathbf{i} + \frac{\Gamma}{2\pi r}(-\cos\theta, \sin\theta).$$

For large r the velocity decays like $1/r$ to the uniform flow $(U, 0)$.

For incompressible flow the density is constant, and, ignoring changes in the external potential F, Bernoulli's equation takes the form

$$\frac{1}{2}u^2 + \frac{p}{\rho} = \frac{1}{2}u_\infty^2 + \frac{p_\infty}{\rho}.$$

Solving for p we obtain

$$p = p_\infty + \frac{\rho}{2}(U^2 - u^2). \tag{4.8}$$

The force exerted on an obstacle with finite boundary σ by the steady flow of fluid around it is

$$f_i = -\iint_\sigma p\nu^i \, dS,$$

where ν is the outward normal. There are no forces due to the momentum, since the normal component of the velocity vanishes on σ. Let Σ be any larger surface containing σ in the interior. By combining the two equations in (4.5) we have, for steady flow in the absence of external forces,

$$\frac{\partial p}{\partial x^i} + \frac{\partial(\rho u^j u_i)}{\partial x^j} = 0, \qquad i = 1,\, 2,\, 3.$$

Integrating this expression over the region between σ and Σ, and using the divergence theorem, we find

$$f_i = -\iint_\sigma p\nu^i \, dS = -\iint_\Sigma p\nu^i + (\rho u_i)u_\nu \, dS. \tag{4.9}$$

Note that $u_\nu = u \cdot \widehat{\nu}$ vanishes on σ but not on Σ, since the latter is not the boundary of an object.

Take Σ to be the circle of radius R centered at the origin, with R so large that σ is contained within. Then $\nu = (\cos\theta, \sin\theta)$, $u_\nu = U\cos\theta$, and the second integral in (4.9) becomes

$$f_i = -\int_0^{2\pi} (p\nu^i + \rho u_i U \cos\theta) R d\theta.$$

These integrals are calculated as $R \to \infty$, using the asymptotic behavior of the flow velocity given above. These considerations lead to

$$f_1 = 0, \qquad f_2 = -\rho\Gamma U$$

These results were obtained independently by KUTTA in 1910 and JOUKOWSKI in 1906. In this model, the actual force on the object is unaffected by its shape, since only the asymptotic behavior of the flows at infinity determines the force; but *the effects of viscosity have been ignored.*

Due to viscosity, the fluid must adhere to the surface of an object, so that the flow field vanishes on the boundary. When the viscosity is small, as it is for air and water, its effects are very small away from the boundary, and are confined to a thin layer near the boundary, called the *boundary layer*. Classical fluid mechanics deals largely with inviscid irrotational flow, for which the powerful tool of potential theory is applicable. The modern theory of fluid mechanics deals with the complicated effects of viscosity and the resulting turbulence within the boundary layer. An extensive treatment of the relation of viscosity and boundary layer theory to potential flow is given in the text by BATCHELOR.

The calculation above shows the fundamental importance of circulation to the theory of forces applied to an airfoil or sail. In the idealized case there is lift but no drag; but a mechanism is needed to create the circulation, and that mechanism is the viscosity of the fluid and the resulting boundary layer around the wing. The optimum design of an airfoil seeks to determine a shape which will generate just the correct amount of circulation in the flow. The correct circulation is that for which the stagnation point sits at the trailing edge of the airfoil; this is known as the Joukowski hypothesis of airfoil design.

The flows with finite circulation are given by (2.21) in the exterior of the unit disk. The flow in the exterior of an airfoil is obtained from the conformal mapping of the exterior of the unit disk to the exterior of the airfoil. In Joukowski's theory of the airfoil, a conformal mapping is sought which maps the rearward stagnation point onto the trailing cusp of the airfoil.

4.3 D'Alembert's Paradox

It was proposed by D'Alembert in 1768 that the force on an object in a three dimensional flow should be zero. This result, known as D'Alembert's paradox,[6] can be proved mathematically as a consequence of the fundamental difference in the asymptotic behavior of harmonic functions in exterior domains in two and three dimensions. In two dimensions, the irrotational flow $(-y, x)r^{-2}$ decays like r^{-1} at infinity. It is the gradient of the multiple-valued harmonic function $\theta(x, y) = \tan^{-1} y/x$. We have seen above that it is precisely this term that generates the force on an object in the flow.

There is no such term, however, for flows in exterior domains in $\mathbb{R}^3$. The motion of an incompressible, irrotational fluid in the exterior of a domain σ in $\mathbb{R}^3$ is obtained as the gradient of a function φ which is harmonic in the exterior of σ and satisfies $\varphi_\nu = 0$ on σ. In order that the flow be asymptotic to a uniform stream with velocity $U\mathbf{i}$ (here $\mathbf{i}$ denotes the unit vector in the x_1 direction) at infinity, we again require $\varphi \sim Ux_1$ as $r \to \infty$. In three dimensions, however, this asymptotic limit is approached like r^{-2}.

Lemma 4.3.1 *Let φ be harmonic in the exterior of σ, $\varphi_\nu = 0$ on σ, and suppose that $\varphi \sim Ux_1$ as $r \to \infty$. Then*

$$\varphi = Ux_1 + O(r^{-2}), \qquad \mathbf{u} = \nabla\varphi = U\mathbf{i} + O(r^{-3}).$$

Proof: Choose coordinates so that the origin is contained within σ, and let Σ be the sphere of radius R centered at the origin, with R large enough that σ lies within Σ. In the exterior of any sphere containing σ the harmonic function φ has an expansion in inverse powers of r, so that

$$\varphi = Ux_1 + \frac{C}{r} + O(r^{-2}).$$

We shall show that $C = 0$. Since φ is harmonic in the region between

[6] J. L. D'Alembert, *Opuscules Mathematiques* **5** (1768). See also the discussion in [21]

σ and Σ, we have

$$0 = \iint_\sigma \frac{\partial \varphi}{\partial \nu} dS = \iint_\Sigma \frac{\partial \varphi}{\partial r} dS = \iint_\Sigma (U\nu_1 - CR^{-2} + O(R^{-3}))R^2 d\omega$$

$$= -4\pi C + O(R^{-1}),$$

since the integral of ν^i over any closed surface vanishes by the divergence theorem. It follows that $C = 0$. The decay of the velocity field then follows by differentiation. ■

It follows that $p = p_\infty + O(r^{-3})$ for flows in $\mathbb{R}^3$. From the integral over Σ in (4.9), we obtain

$$f_i = -\iint_\Sigma \Big((p_\infty + O(R^{-3}))\nu^i + (\rho U\delta_{i1} + O(R^{-3})(U\nu_1 + O(R^{-3})\Big)R^2 d\omega.$$

Again noting that the integral of ν^i over any closed surface vanishes, we see that the above integral behaves like R^{-1}. Letting $R \to \infty$, we see that the force must vanish.

The implication of D'Alembert's paradox is that forces on an airfoil can only be generated by two dimensional flows; this means that conformal mapping techniques are relevant to the discussion of forces on objects in fluid flows, and that three dimensional effects are, to some extent, relative – that three dimensional flows with a small aspect ratio can be approximated by two dimensional flows.

4.4 Hyperbolic Conservation Laws

A partial differential equation is said to be in *divergence* form if it can be written in the general form $A_t + B_x = 0$. Euler's equations (4.5) for one dimensional isentropic flow, may be rewritten in divergence form as follows.

$$\rho_t + (\rho u)_x = 0 \qquad (\rho u)_t + (\rho u^2 + p)_x = 0. \tag{4.10}$$

These equations are supplemented by an equation of state $p = p(\rho)$. Such a system of equations is called a system of *conservation laws.*

Equations (4.10) also form a hyperbolic system, that is, they have two real characteristics. We discussed characteristics of second order scalar equations in Chapter 3. The notion of characteristic for a system is much the same. Consider a general first order system of equations

$$AU_t + BU_x + CU + D = 0, \tag{4.11}$$

where U is a column vector of length n, and $A, \ldots D$ are $n \times n$ matrices depending smoothly on x and t. For example, the system (4.10) can be written in the form

$$AU_t + BU_x = 0, \qquad A = \begin{pmatrix} 1 & 0 \\ u & \rho \end{pmatrix}, \quad B = \begin{pmatrix} u & \rho \\ c^2 + u^2 & 2\rho u \end{pmatrix}, \quad U = \begin{pmatrix} \rho \\ u \end{pmatrix}, \tag{4.12}$$

where $c^2 = p'(\rho)$. The quantity $c(\rho)$ is the speed of sound.

Given a curve Γ in the $x - t$ plane, we ask "When are the values of U on Γ compatible with (4.11)?". Or, to put the question another way, when are the values of U on Γ and the equation (4.11) overdetermined? If this is the case, then we say that Γ is a characteristic. To answer this, let Γ be parameterized by smooth functions $x(s)$, $t(s)$. If U is specified on Γ, then its tangential derivative U_τ along Γ is given by

$$U_\tau = \frac{dU}{ds} = U_x x' + U_t t'.$$

Moreover, the differential equation also gives a relationship between U_x and U_t, so that we have a system of $2n$ equations in the $2n$ unknowns U_x and U_t

$$AU_t + BU_x = f_1(s), \qquad U_x x' + U_t t' = f_2,$$

where f_1 and f_2 are entirely determined by the values of U on Γ and (4.11).

By definition, Γ is a characteristic when these equations are not uniquely solvable. This is equivalent to the statement that

$$\begin{pmatrix} x'I & t'I \\ B & A \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 0$$

has a non-trivial solution. This reduces to the system $(x'A - t'B)U_2 = 0$. Thus Γ is a characteristic if $\det(B - \dot{x}A) = 0$.

The system (4.11) is said to be *hyperbolic* if $\det(B - \lambda A)$ has n real roots $\lambda_1, \ldots, \lambda_n$. The roots are the characteristic speeds and the associated eigenvectors are the characteristic directions. In the simple linear case $u_t + v_x = 0$, $v_t + u_x = 0$ the characteristics are $x \pm t = const.$

The same concepts apply in the nonlinear case when A and B are functions of the dependent variables, as in the case of (4.10). In the nonlinear case, however, the characteristic speeds and directions depend on the solution. We leave it as an exercise to show that the characteristic speeds of the hyperbolic system (4.12) are $\dot{x} = u \pm c$.

For the first order hyperbolic system $u_t + v_x = 0$, $v_t + u_x = 0$ the functions $u \pm v$ are constant along the characteristics. For example, differentiating $u + v$ along a curve $dx/dt = 1$, we find

$$\frac{d}{dt}(u+v) = (u+v)_t + (u+v)_x \frac{dx}{dt} = 0.$$

Similar invariants, called Riemann invariants, exist for nonlinear hyperbolic systems as well. For example, in (4.10) we look for functions of ρ and u which are invariant along the characteristic curves $dx/dt = u \pm c(\rho)$. Let us denote by $r(\rho, u)$ a function which is constant along the curve $dx/dt = u + c$. By the chain rule

$$\frac{d\rho}{dt} = \rho_t + \rho_x \frac{dx}{dt} = \rho_t + \rho_x(u+c), \qquad \frac{du}{dt} = u_t + u_x \frac{dx}{dt} = u_t + u_x(u+c),$$

so

$$\begin{aligned} \frac{dr}{dt} =& r_\rho \frac{d\rho}{dt} + r_u \frac{du}{dt} = r_\rho(\rho_t + (u+c)\rho_x) + r_u(u_t + (u+c)u_x) \\ =& r_\rho((u+c)\rho_x - (\rho u)_x) + r_u((u+c)u_x - uu_x - \frac{c^2}{\rho}\rho_x) \\ =& \left(r_\rho - \frac{c}{\rho} r_u \right)(c\rho_x - \rho u_x). \end{aligned}$$

Thus, $r = r(\rho, u)$ is constant along the characteristics $\dot{x} = u + c$ if

$$r_u = \frac{\rho}{c} r_\rho.$$

It is easily seen that we may take

$$r(\rho, u) = \frac{1}{2}\left(u + \int^{\rho} \frac{c(\rho')}{\rho'}\, d\rho' \right).$$

Similarly,

$$s(\rho, u) = \frac{1}{2}\left(u - \int^{\rho} \frac{\rho'}{c(\rho')}\, d\rho'\right)$$

is invariant along the characteristics curves $\dot{x} = u - c$. When $p = N\rho^{\gamma}$, the integrations may be carried out explicitly, and we obtain

$$r(\rho, u) = \frac{1}{2}\left(u + \frac{c}{\gamma - 1}\right), \qquad s(\rho, u) = \frac{1}{2}\left(u - \frac{c}{\gamma - 1}\right).$$

These are the Riemann invariants for an ideal gas.

4.5 Shocks

These arguments are valid whenever the solutions are in C^1; but they lead directly to the conclusion that discontinuities in the solutions may form after a finite time, no matter how smooth the initial data. For example, suppose the initial data are such that $s(x, 0) \equiv$ *const.* on some interval I on the $x-$ axis. Then s is constant on the domain of influence of I in the region $t > 0$, so long as the solutions remain smooth. Since the other Riemann invariant, r, is constant on the curves $\dot{x} = u + c$, both u and ρ are constant along these characteristics, and they are therefore straight lines. If now $u + c$ is initially decreasing in x on some interval, these straight lines must intersect at some point in the region $t > 0$. At this point of intersection the values of ρ and u must necessarily be different; hence at such a point the solutions must have a discontinuity. Such discontinuities are called *shocks*, and play a fundamental role in the analysis of the equations of gas dynamics. We give here only a brief introduction to this very extensive subject. For further details, the reader should see the texts by Courant and Friedrichs [6], Landau and Lifshitz [12], Serrin [21], Smoller [22], Whitham [26].

Let us turn to a discussion of the hyperbolic equations (4.10) in the class of solutions which are not C^1. First consider the case where the solutions are continuous in a domain Ω, with a discontinuity in their derivatives along a differentiable curve γ, given by $x = x(t)$. Let $f(x, t)$ be any function defined in a neighborhood of γ, and denote the limiting values of f and its derivatives on γ from the right ($x > x(t)$) and left ($x < x(t)$) by $f^{\pm}, f_x^{\pm}$, and $f_t^{\pm}$ respectively. If $[f] = f^+ - f^-$, denotes the jump of f across the curve, etc, then the continuity of f across γ is equivalent to the statement that $[f] = 0$; while jumps in f_x and f_t imply that $[f_x]$, $[f_t]$ are non zero.

The derivative of f along the curve is $df(x(t),t) = f_t + f_x\dot{x}$. Since $[f] = 0$ on the curve,

$$\frac{d}{dt}[f] = \left[\frac{df(x(t),t)}{dt}\right] = [f_t + f_x\dot{x}] = [f_t] + \dot{x}[f_x] = 0.$$

Applying this jump relation to the variables ρ and u, we have

$$[\rho_t] + \dot{x}[\rho_x] = 0, \qquad [u_t] + \dot{x}[u_x] = 0.$$

On the other hand, from (4.5) we have

$$[\rho_t] + [(\rho u)_x] = 0, \qquad [u_t] + u[u_x] + \frac{c^2(\rho)}{\rho}[\rho_x] = 0.$$

Using the jump relations we may eliminate the jumps in the time derivatives from the equations of motion, thus obtaining

$$(\dot{x} - u)[\rho_x] = \rho[u_x], \qquad (u - \dot{x})[u_x] + c^2\frac{[u_x]}{(\dot{x} - u)} = 0.$$

If $[u_x] \neq 0$ these equations imply that the speed of the curve γ, that is, the speed with which the discontinuity propagates, is $\dot{x} = u \pm c$. As we have seen, these are the characteristic speeds of the Euler equations. Thus, a jump in the derivative of either u or ρ implies a jump in the other, as well as a jump in the pressure gradient p_x. For this reason the discontinuity is interpreted as a sound wave.[7]

Now let us turn to an analysis of the equations (4.10) in a neighborhood of a discontinuity in the flow variables. Recall that the Euler equations are expressions of the basic conservation laws in differential form. We derive the conservation laws across a shock discontinuity.

For discontinuous solutions the partial differential equations must be reformulated in weak form

$$\iint_\Omega \rho\,\varphi_{1,t} + \rho u\,\varphi_{1,x}\,dxdt = 0, \qquad \iint_\Omega \rho u\,\varphi_{2,t} + (\rho u^2 + p)\varphi_{2,x}\,dxdt = 0.$$

[7] This derivation of the speed of sound is due to HUGONIOT in 1885-1888; see the discussion in [21], p. 212.

These equations are to hold for all functions $\varphi_1, \varphi_2 \in C_0^1(\Omega)$, where Ω is an open region in the $x - t$ plane. As usual, any weak solution which is regular in any subdomain is necessarily a strong solution in that subdomain.

The following is used to derive the jump conditions across a shock.

Lemma 4.5.1 *Suppose that F and G are piecewise C^1 functions in a domain Ω in the x-t plane, with jump discontinuities across a smooth curve Γ given by $x = x(t)$. Let $\Omega_\pm$ denote the components of Ω on the left and right of the curve Γ, oriented in the direction of increasing time. Let F and G satisfy*

$$\iint_\Omega F\varphi_t + G\,\varphi_x\,dxdt = 0, \qquad \forall\ \varphi \in C_0^1(\Omega).$$

Then

$$[G] = [F]\dot{x}$$

where $[F]$ and $[G]$ denote the jumps of F and G across Γ.

Proof: By the usual argument, $F_t + G_x = 0$ in each of the subdomains $\Omega_\pm$. By Green's theorem,

$$\oint_{\partial\Omega_\pm} G\varphi\,dt - F\varphi\,dx = \iint_{\Omega_\pm} (G\varphi)_x + (F\varphi)_t\,dxdt.$$

The line integrals are oriented so that Ω_+ and Ω_- lie on the left of the path. Since φ vanishes on $\partial\Omega$ the only contribution to the line integrals is along the curve Γ. Adding the two equations above we get

$$\int_\Gamma \varphi([F]dx - [G]dt) = \iint_{\Omega_+\cup\Omega_-} (G\varphi)_x + (F\varphi)_t\,dxdt$$

$$= \iint_\Omega (F_t + G_x)\varphi + F\varphi_t + G\varphi_x\,dxdt.$$

The first term in the double integral vanishes since $F_t + G_x = 0$ in Ω_+ and Ω_-; and the second integral vanishes by hypothesis. Since φ is an arbitrary smooth function, $[G]dt - [F]dx$ must vanish along Γ, and the lemma is proved. ■

When we apply this argument to the equations (4.10), we obtain

$$[\rho u] = [\rho]\dot{x}, \qquad [\rho u^2 + p] = [\rho u]\dot{x}.$$

Letting $U = u - \dot{x}$ be the flow velocity relative to the shock, the above equations can be rewritten in the form

$$[\rho U] = 0, \qquad [\rho u U + p] = 0. \tag{4.13}$$

These two jump relations express conservation of mass and momentum across a shock, just as the Euler equations express these conservation laws in regions where the flow is regular.

Caveat: The divergence form of the conservation laws is in general not unique (cf. exercise 8); hence one must be careful in choosing the weak form of the equations. The various weak forms of the equations are not equivalent, and will lead to different conservation laws across the shocks. Hence it is essential to choose the physically correct divergence form for the equations. The advantage of the weak formulation of the equations is that it contains all the information about the solutions, including the shock conditions, and remains valid even when the shock structure is extremely complicated. An alternative derivation of the shock conditions can be given directly by extending the transport theorem to cases in which a shock occurs in the interior of the domain Ω.

Given the state before the shock and the speed of the shock, that is, given $\rho_+,\ p_+,\ U_+$ it is not possible to determine $\rho_-,\ p_-,\ U_-$ from these two equations alone, since they constitute two equations in three unknowns. For, since we cannot assume that the entropy is constant across the shock, we no longer know the relationship between ρ and p behind the shock. Thus the resolution of the problem when there are shocks requires the introduction of thermodynamic considerations [21].

4.6 Exercises

1. A *stagnation point* is a point at which the velocity vanishes. Find the stagnation points of the flow given by (2.21) for all values of $\Gamma/4\pi U$. Hint: The velocity in polar coordinates is given by

$$v_r = \frac{\partial \varphi}{\partial r}, \qquad v_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial r}.$$

2. The velocity in an irrotational flow cannot attain a maximum value in the interior of the domain of the flow; the pressure cannot attain an interior minimum.

3. The linearized Euler equations in three dimensions for small disturbances about the rest state $u = 0,\ p = p_0,\ \rho = \rho_0$ are obtained by neglecting quadratic terms. By eliminating u show that the pressure satisfies the wave equation

$$p_{tt} = c^2 \Delta p, \qquad c^2 = p'(\rho_0).$$

4. Prove the following extension of the transport theorem: Let f be a scalar quantity, v a vector field, and $\Omega(t)$ a domain in $\mathbb{R}^3$ moving with the flow generated by v. Suppose f is piecewise differentiable in Ω with jump $[f]$ across a smooth surface $\Gamma = \Gamma(t, s)$, (s=parameter) contained in $\Omega(t)$. Then

$$\frac{d}{dt} \iiint\limits_{\Omega} f\, dx = \iiint\limits_{\Omega} \frac{df}{dt}\, dx + \iint\limits_{\Gamma} [fU] dS$$

where $U = (v - \Gamma_t) \cdot \hat{\nu}_\Gamma$ is the relative normal velocity of the fluid across Γ.

Use this form of the transport equation to derive the conservation laws across a shock.

5. Find the characteristic speeds for Euler's equations

$$\rho_t + (\rho u)_x = 0, \qquad \rho u_t + \rho u u_x + c^2 \rho_x = 0.$$

Show that for strong solutions they are equivalent to the equations in divergence form (4.10). Show the second equation may also be written in the divergence form

$$u_t + \left(\frac{u^2}{2} + \int \frac{c^2}{\rho} \right)_x = 0.$$

What is the conservation law across the shock corresponding to this divergence form of the momentum equation?

Chapter 5

The Maximum Principle

In this chapter we prove the strong maximum principle for second order elliptic operators and state, but do not prove, the corresponding result for general parabolic operators. Maximum principles provide a unique, powerful tool, for scalar elliptic and parabolic operators of second order, and we shall illustrate some of the many applications later in the chapter.

5.1 Elliptic and parabolic inequalities

Throughout this section, we let L denote the following second order differential operator:

$$Lu = \sum_{j,k=1}^{n} a_{jk}(x)u_{jk} + \sum_{j=1}^{n} b_j(x)u_j,$$

where

$$u_j = \frac{\partial u}{\partial x_j}, \qquad u_{jk} = \frac{\partial^2 u}{\partial x_j \partial x_k}.$$

The operator L is said to be uniformly elliptic in a domain Ω if there is a positive constant μ such that

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \geq \mu \sum_{j=1}^{n} \xi_j^2, \qquad \forall x \in \Omega.$$

We assume throughout that Ω is a connected open set in $\mathbb{R}^n$.

We begin by proving the weak maximum principle.

Theorem 5.1.1 *Assume L is uniformly elliptic and that its coefficients a_{jk}, b_j are continuous and uniformly bounded in Ω. If $u \in C^2(\Omega)$ and*

$$Lu > 0, \qquad x \in \Omega,$$

then u cannot have an interior maximum in Ω.

Proof: At an interior maximum, all the first derivatives of u vanish, and the matrix u_{jk} of second derivatives is non-positive. Thus, at an interior maximum,

$$Lu = \mathrm{Tr}\, au, \qquad a = ||a_{jk}||, \qquad u = ||u_{jk}||.$$

Since a is positive definite and u is non-positive, the trace of their product cannot be positive, hence $Lu \leq 0$ at an interior maximum. ■

We use this result to prove the following, which is known as the strong maximum principle of E. Hopf. It generalizes the result for Laplace's equation to uniformly elliptic, second order equations. We state it here as a one sided inequality; this gives a more general result. If $Lu \geq 0$ in Ω, then u cannot attain an interior maximum in Ω; and if $Lu = 0$ in Ω, then u cannot attain an interior extremum.

Theorem 5.1.2 *Assume L is uniformly elliptic in a domain Ω and that its coefficients a_{jk}, b_j are continuous and uniformly bounded in Ω. If $u \in C^2(\Omega)$ and*

$$Lu \geq 0, \qquad x \in \Omega,$$

then u cannot have an interior maximum in Ω unless u is identically constant.

Proof: Let M be the supremum of u over Ω, and let

$$\Omega_M = \{x \in \Omega \,:\, u(x) < M\}.$$

Since u is continuous, Ω_M is open. We are going to show that Ω_M is also closed in the relative topology of Ω. That is, if B is an open ball contained in Ω_M and $u(p) = M$, where $p \in \partial B$, then $p \in \partial\Omega$. It then follows that either Ω_M is empty, in which case $u \equiv M$, or $\Omega_M = \Omega$.

Let $S = \partial B$ and let $p \in S$, $u(p) = M$. We shall show that

$$\frac{\partial u}{\partial \nu}(p) > 0, \tag{5.1}$$

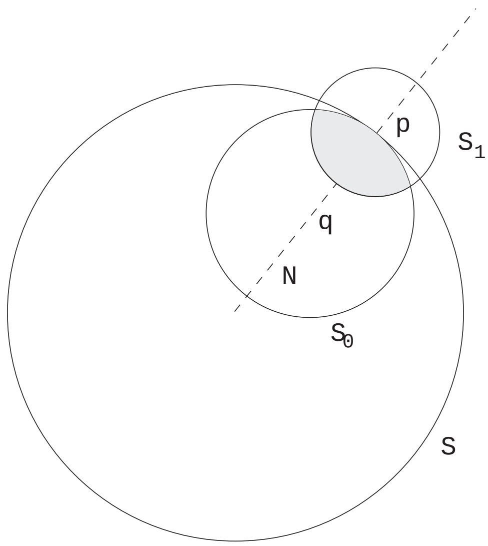


Figure 5.1: Proof of the boundary point lemma.

where ν is the outward normal to S at p. It then follows that $p \in \partial\Omega$, since all first order derivatives of u must vanish at an interior maximum.

Let N be the normal line to S through p, and choose a point $q \in B$ on N such that $|p - q| = r_0$, where r_0 is less than the radius of S. We take q as the origin and consider the function

$$h(r) = e^{-\alpha r^2} - e^{-\alpha r_0^2}.$$

It is clear that $h = 0$ on the sphere S_0 of radius r_0 centered at q, and $h > 0$ in the interior of S_0. Moreover,

$$Lh = e^{-\alpha r^2}\left(4\alpha^2 \sum_{j,k=1}^{n} a_{jk}x_jx_k - 2\alpha \sum_{j=1}^{n}(a_{jj} + b_jx_j)\right);$$

so $Lh > 0$ in a neighborhood of q for sufficiently large α.

Let S_1 be a sphere centered at p, and let C be the intersection of the interiors of S_0 and S_1, indicated by the shaded region in Figure (5.1). Put $v = u + \varepsilon h$; then

$$Lv = Lu + \varepsilon Lh \geq \varepsilon Lh > 0, \qquad x \in C.$$

By the weak maximum principle, the maximum of v must occur on the boundary of C. Since $h = 0$ on S_0, $v = u$ on S_0, and $v = M$ at p. The second component of ∂C is a circular arc lying strictly in the interior of S. On this arc, u is bounded away from M, so for sufficiently small ε, $v = u + \varepsilon h < M$.

Hence the maximum of v in $\bar{C}$ is M, and this value is attained at p, and p only. At that point we must have

$$\frac{\partial v}{\partial \nu}(p) \geq 0,$$

where ν is the outward normal to S at p. Since $\partial h/\partial \nu(p) < 0$, (5.1) follows. ■

The key point in this proof is the inequality (5.1); this result is called the boundary point lemma. We have shown the following:

Theorem 5.1.3 [Boundary Point Lemma] *Suppose the conditions of Theorem* (5.1.2) *hold on a domain Ω. Suppose that u is not identically constant on Ω and attains its maximum at $p \in \partial\Omega$. Suppose that there is a sphere S whose interior lies in Ω and which is tangent to $\partial\Omega$ at p. Then*

$$\frac{\partial u}{\partial \nu}(p) > 0, \tag{5.2}$$

where ν is the outward normal to Ω at p.

There are several extensions of the maximum principle worth noting.

Theorem 5.1.4 *Let the previous assumptions on L hold, $c \leq 0$ in Ω, and*

$$Lu + cu \geq 0, \qquad x \in \Omega.$$

Then u cannot attain a non-negative interior maximum in Ω unless u is identically constant in Ω. Moreover, if $u < 0$ in Ω and $u(p) = 0$ for some point $p \in \Omega$ at which the sphere condition holds, then (5.2) *holds.*

Proof: The proof of this theorem proceeds exactly as the proof of the strong maximum principle. In this case the parameter α must be chosen so that $(L + c)h > 0$. This is certainly possible when c is bounded, since $0 \leq h \leq 1$.

Corollary 5.1.5 *Let the previous assumptions on L hold, and let $c \leq 0$ in Ω. If*

$$Lu + cu = 0, \qquad x \in \Omega,$$

then u cannot attain an interior maximum or minimum in Ω unless u is identically constant on Ω.

We leave the proof of this result to the reader.

Theorem 5.1.6 *Let the previous assumptions on L hold, and let $c(x)$ be a bounded function (not necessarily continuous) on Ω. Suppose that $u \in C^2(\Omega)$ and*

$$(L + c)u \geq 0, \qquad u \leq 0 \qquad x \in \Omega.$$

Then either $u \equiv 0$ on Ω or $u < 0$ in Ω.

Moreover, if $u < 0$ in Ω then (5.2) holds at any $p \in \partial\Omega$ at which $u(p) = 0$ and the sphere condition is satisfied.

Note that there are no restrictions on the sign of c.

Proof: Write $c = c_+(x) + c_-(x)$, where $c_+ \geq 0$ and $c_- \leq 0$ in Ω; and write the inequality as

$$(L + c_-)u \geq -c_+ u \geq 0, \qquad x \in \Omega.$$

Let $N = \{u < 0\}$. By the argument used in the proof of Theorem (5.1.2), N is open by continuity of u, and closed in the relative topology of Ω as a consequence of the boundary point lemma. Therefore N is either empty or all of Ω. ∎

The maximum principle can be extended to scalar second order parabolic operators by much the same argument. We continue to make the same assumptions about the elliptic operator L defined above, except that now we allow the coefficients a_{jk} and b_j to depend on x and t. Let $\Omega_\tau = \Omega \times [0, \tau]$ for any $\tau > 0$.

Theorem 5.1.7 *Let $u(x, t)$ satisfy the parabolic differential inequality*

$$Lu - \frac{\partial u}{\partial t} \geq 0, \qquad (x, t) \in \Omega_T.$$

Suppose that u attains an interior maximum M at a point (p, t_0), where $p \in \Omega$ and $0 < t_0 \leq T$. Then u is identically equal to M in the cylinder $\overline{\Omega_{t_0}}$.

Suppose that u attains its maximum at a point (p, t_0) for some $p \in \partial\Omega$, and that a sphere S_p tangent to $\partial\Omega$ at p can be constructed in Ω. Then

$$\frac{\partial u}{\partial \nu}(p) > 0,$$

where ν is the outward normal to Ω at p.

Remark: The maximum of u must occur either on the sides of the cylinder, i.e. on $\partial\Omega \times [0, T]$ or in Ω at time $t = 0$. Physically, if u denotes the temperature and it is below freezing outside Ω and below freezing initially, then it is never going to get above freezing inside Ω.

The maximum principle for parabolic operators can be extended as follows.

Theorem 5.1.8 *Let h be continuous in Ω_T and bounded above, and suppose that*

$$Lu + hu - u_t \geq 0, \qquad (x, t) \in \Omega_T, \qquad Bu \leq 0.$$

Then either $u \equiv 0$ or $u < 0$ in Ω_T.

Proof Put $u = ve^{\lambda t}$; then

$$Lv + (h - \lambda)v - v_t \geq 0, \qquad Bv \leq 0.$$

We may choose λ so that $h - \lambda < 0$ in Ω_T, since h is bounded above. If $u \geq 0$ in the interior of Ω_T then $v \geq 0$ somewhere also, and v must have a non-negative maximum somewhere in Ω_T. At such a non-negative maximum of v we have $v_t \geq 0$ and $Lv \leq 0$, but

$$Lv - v_t \geq (\lambda - h)v \geq 0.$$

By continuity, this inequality must hold in an open set; hence the conclusion of the theorem follows from the strong maximum principle, Theorem (5.1.7). ■

For a full discussion of maximum principles and their applications to partial differential equations, see the monograph by Protter & Weinberger [15].

5.2 Monotone methods

Maximum principles have myriad applications in partial differential equations. Of course, they can be used to prove uniqueness theorems, but I will begin with a simple application to semi-linear equations [17]. Consider the simple nonlinear boundary value problem

$$Lu + f(x,u) = 0, \qquad x \in \Omega \tag{5.3}$$

$$Bu = g, \qquad Bu = u\Big|_{\partial\Omega} \tag{5.4}$$

where L is a uniformly elliptic second order operator with Hölder continuous coefficients as defined in the previous section. Remark: We require Hölder continuity of the coefficients so that the solution of the linear equation $Lu = f$ is smooth. We assume that $\partial\Omega$ is smooth. An *upper solution* for (5.3) (5.4) is a C^2 function $\bar{v}$ which satisfies the inequalities

$$L\bar{v} + f(x,\bar{v}) \leq 0, \qquad B\bar{v} \geq g.$$

A *lower solution* $\underline{v}$ is defined by reversing the inequalities. The following theorem was obtained by H. Amman [3] and D.H. Sattinger [17]; cf. also [4] and [18].

Theorem 5.2.1 *Let* $\underline{v} \leq \bar{v}$ *be lower and upper solutions for the boundary value problem* (5.3), (5.4). *Then there exists at least one solution* u *of the nonlinear boundary value problem such that* $\underline{v} \leq u \leq \bar{v}$.

Proof: The solution is obtained by a monotone iteration scheme as follows. Let A be a positive real number, chosen so that

$$f_A(x,u) = f(x,u) + Au$$

is an increasing function of u on the interval $[\min \underline{v}, \max \bar{v}]$. Let $u^0 = \bar{v}$ and solve the linear boundary value problem

$$(L - A)u^1 + f_A(x,u^0) = 0, \qquad x \in \Omega; \qquad Bu^1 = g.$$

Note that

$$(L - A)(u^1 - u^0) = -\left(Lu^0 + f(x,u^0)\right) \geq 0, \qquad x \in \Omega$$

$$B(u^1 - u^0) = g - Bu^0 \leq 0.$$

Therefore, by the maximum principle, (extended to the operator $L-A$), the function $u^1 - u^0$ cannot have a positive interior maximum. Since it is non-positive on the boundary, $u^1 < u^0 = \bar{v}$ everywhere in the interior of Ω by the strong maximum principle. It follows that

$$Lu^1 + f(x,u^1) = (L-A)u^1 + f_A(x,u^1) = f_A(x,u^1) - f_A(x,u^0) \leq 0,$$

since $u^1 < u^0$ and f_A is increasing. Therefore u^1 is also an upper solution.

Continuing in this way we obtain a decreasing sequence of upper solutions

$$\bar{v} = u^0 > u^1 > u^2 > \dots \quad .$$

Similarly, starting with $u_0 = \underline{v}$ we obtain an increasing sequence of lower solutions:

$$\underline{v} = u_0 < u_1 < u_2 < \dots \quad .$$

We leave it as an exercise to show that $u_j < u^k$ for any $j,\ k$. Therefore each of the decreasing sequence of upper solutions is bounded below by each member of the increasing sequence of lower solutions. By the standard regularity theory for elliptic partial differential equations, one can show that each of these sequences converges, and that we obtain solutions (which may be distinct)

$$\underline{v} \leq \underline{u} \leq \bar{u} \leq \bar{v}. \quad \blacksquare$$

We can also use the maximum principle to investigate stability of solutions of the initial value problem for the associated parabolic equation:

$$Lu + f(x,u) - u_t = 0, \qquad Bu = g, \qquad u(x,0) = u_0(x). \tag{5.5}$$

Lemma 5.2.2 *Let $\underline{v}$, $\bar{v}$ be C^2 functions which satisfy the inequalities*

$$\begin{aligned} L\bar{v} + f(x,\bar{v}) - \bar{v}_t &\leq 0 \\ L\underline{v} + f(x,\underline{v}) - \underline{v}_t &\geq 0 \end{aligned}$$

in the domain Ω_T, and

$$\underline{v} \leq \bar{v} \qquad on \ \ B_T,$$

where $B_T = (\Omega \times \{t=0\} \cup (\partial\Omega \times [0,T])$. Then

$$\underline{v}(x,t) < \bar{v}(x,t) \qquad (x,t) \in \Omega_T.$$

Proof: Let $w = \underline{v} - \bar{v}$; then

$$Lw + [f(x, \underline{v}) - f(x, \bar{v})] - w_t \geq 0, \qquad (x, t) \in \Omega_T,$$
$$w \leq 0 \qquad \text{on } B_T.$$

By the mean value theorem,

$$\begin{aligned} f(x, \underline{v}) - f(x, \bar{v}) &= \int_{\bar{v}}^{\underline{v}} f_u(x, s) ds \\ &= \int_0^1 f_u(x, \bar{v} + zw) w \, dz = F(x, w) w. \end{aligned}$$

Therefore, $w = w(x, t)$ satisfies the parabolic differential inequality

$$Lw + Fw - w_t \geq 0, \qquad Bw \geq 0.$$

We do not know $F = F(x, w)$ explicitly, but we may regard it as a smooth coefficient in the above inequality. Since w is a smooth bounded function, it follows that F is bounded, and the lemma follows by Theorem 5.1.8. ■

Corollary 5.2.3 *Let $\underline{v} \leq u_0 \leq \bar{v}$, where $\underline{v}$ and $\bar{v}$ are lower and upper solutions of* (5.3), (5.4)*, and let u satisfy the initial value problem* (5.5)*. Then*

$$\underline{v} \leq u(x, t) \leq \bar{v}, \qquad \forall \ t > 0.$$

Proof: Since $\underline{v}$ and $\bar{v}$ are time independent, they satisfy the parabolic inequalities of Lemma (5.2.2), and the Corollary follows.

Theorem 5.2.4 *Let $\underline{v}_0 < \bar{v}_0$ be lower and upper solutions of* (5.3), (5.4)*, and let $\bar{v}(x, t)$ satisfy the initial value problem*

$$L\bar{v} + f(x, \bar{v}) - \bar{v}_t = 0, \qquad \bar{v}(x, 0) = \bar{v}_0(x), \qquad B\bar{v}(\cdot, t) = g(\cdot).$$

Then $\bar{v}_t(x, t) < 0$ for $t > 0$. If $\underline{v}(x, t)$ satisfies the initial value problem and $\underline{v}(x, 0) = \underline{v}_0(x)$, then $\underline{v}_t > 0$ for $t > 0$. Moreover, $\underline{v}(x, t) < \bar{v}(x, t)$ for all $t > 0$.

The solution $\underline{v}(x, t)$ increases monotonically to a stationary solution $\underline{U}(x)$ of (5.3), (5.4)*; while $\bar{v}(x, t)$ decreases monotonically to a stationary solution $\bar{U}(x)$; and $\underline{U}(x) \leq \bar{U}(x)$.*

Proof: Let $w = \bar{v}_t$. Then w satisfies the parabolic equation

$$Lw + f_u(x, \bar{v})w - w_t = 0, \qquad Bw = 0.$$

Moreover,

$$w(x, 0) = \bar{v}_t(x, 0) = L\bar{v} + f(x, \bar{v}) \leq 0.$$

Therefore $w(x, t) = \bar{v}_t(x, t) < 0$ by Theorem (5.1.8).

The same argument applies to the solution of the initial value problem when the initial data is a lower solution. The inequality $\underline{v}(x, t) < \bar{v}(x, t)$ is a consequence of Theorem (5.1.8). Let $w = \bar{v}(x, t) - \underline{v}(x, t)$. By the mean value theorem, w satisfies a parabolic equation

$$Lw + F(x, t, w)w - w_t = 0, \qquad Bw = 0.$$

Since $w(x, 0) > 0$, it follows that $w(x, t) > 0$ for all $t > 0$.

Since the solutions of the two initial value problems are monotone in t and bounded, their limits exist as $t \to \infty$. Suppose that

$$\bar{U}(x) = \lim_{t\to\infty} \bar{v}(x, t).$$

Let $\varphi = \varphi(x)$ be any smooth $C_0^2(\Omega)$ test function. Then

$$(L\bar{v}, \varphi) + (f(x, \bar{v}), \varphi) - (\bar{v}_t, \varphi) = 0,$$

$$(\bar{v}, L^*\varphi) + (f(x, \bar{v}), \varphi) - (\bar{v}_t, \varphi) = 0,$$

where L^* is the adjoint of the elliptic operator L.

We have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \bar{v}(x, t)dt = \bar{U}(x);$$

hence

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\bar{v}_t, \varphi)dt = \lim_{T\to\infty} \frac{(\bar{v}(x, t)\varphi) - (\bar{v}, \varphi)}{T} = 0$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\bar{v}, L^*\varphi)dt = (\bar{U}, L^*\varphi)$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (f(x, \bar{v}), \varphi)dt = (f(x, \bar{U}), \varphi).$$

Hence $\bar{U}$ is a weak solution of the nonlinear boundary value problem (5.3) (5.4):

$$(\bar{U}, L^*\varphi) + (f(x, \bar{U}), \varphi) = 0, \qquad B\bar{U} = g.$$

We have seen that weak solutions of the Dirichlet problem for the Laplacian ($L = \Delta$) are strong solutions, and the same is true of the more general uniformly elliptic boundary value problem when the coefficients of L are Hölder continuous. ■

A *point of tangency* $p \in \bar{\Omega}$ for two functions $u,\ v \in C^2(\Omega)$ is a point at which the tangent planes to the graphs of the two functions coincide. That is, $u(p) = v(p)$ and $\nabla u(p) = \lambda \nabla v(p)$. When $p \in \partial\Omega$ we define $\nabla u(p)$ to be the limit of $\nabla u(p')$ as $p' \to p$ from the interior of Ω.

Theorem 5.2.5 *Let $\underline{v} \leq \bar{v}$ be lower and upper solutions to* (5.3), (5.4). *If there exists a point of tangency $p \in \bar{\Omega}$, then $\underline{v} \equiv \bar{v}$ throughout $\bar{\Omega}$. In particular, either $\underline{v} \equiv \bar{u}$ or $\underline{v} < \bar{v}$.*

Proof: Let $w = \bar{v} - \underline{v}$; then $w \geq 0$ on $\bar{\Omega}$ and

$$(L - A)w + f_A(x, \bar{v}) - f_A(x, \underline{v}) \leq 0.$$

For sufficiently large $A > 0$, $f_A(x, u)$ is increasing in u; and so

$$(L - A)w \leq 0.$$

Since $w \geq 0$ on $\partial\Omega$, it follows by Theorem (5.1.4) (applied to $-w$) that either $w > 0$ or $w \equiv 0$ in the interior. In particular, there can be no point of tangency in the interior.

Now suppose there is a point $p \in \partial\Omega$ at which $w(p) = 0$. By the maximum principle at the boundary, Theorem (5.1.3), $w_\nu(p) < 0$, hence p cannot be a point of tangency. ■

Corollary 5.2.6 *Let $u_1 \leq u_2$ be two solutions of* (5.3) (5.4). *Then either $u_1 < u_2$ in Ω or $u_1 \equiv u_2$.*

We now discuss a number of examples that show the application of these techniques. It can be shown by a variational argument that the nonlinear elliptic boundary value problem

$$\Delta u = u^2, \qquad x \in \Omega; \qquad Bu = 0 \tag{5.6}$$

has a nontrivial solution; we denote it by w. We assume throughout this discussion that Ω is bounded and that $\partial\Omega$ is a smooth surface in $\mathbb{R}^n$, with the property that at every point $p \in \partial\Omega$ there is a sphere tangent to $\partial\Omega$ that is contained in Ω. By the strong maximum principle, $w < 0$ in Ω. Let us show that w is an unstable equilibrium for the associated parabolic equation $u_t = \Delta u - u^2$.

Consider the function λw. We have

$$\Delta(\lambda w) - (\lambda w)^2 = \lambda(\Delta w - \lambda w^2) = \lambda(1-\lambda)w^2.$$

Hence λw is a lower solution for $\lambda < 0$ or $\lambda > 1$ and an upper solution if $0 < \lambda < 1$. Let $u_\lambda(x,t)$ satisfy the initial value problem

$$\frac{\partial u_\lambda}{\partial t} = \Delta u_\lambda - u_\lambda^2, \qquad Bu_\lambda = 0, \qquad u_\lambda(x,0) = \lambda w(x).$$

By Theorem (5.2.4), $u_\lambda(x,t)$ is decreasing in time if $\lambda > 1$ or $\lambda < 0$ and increasing for $0 < \lambda < 1$.

When $0 < \lambda < 1$, $u_\lambda(x,t)$ increases monotonically to a stable equilibrium. Using Theorem (5.2.5), we prove in the next paragraph that there can be no other equilibrium solution $w_2 > w$. Similarly, when $\lambda < 0$ the solution $u_\lambda(x,t)$ decreases monotonically to zero as $t \to \infty$. Therefore, all solutions of the initial value problem for which the initial data lies above $w(x)$ tend to zero asymptotically as $t \to \infty$. We leave it as an exercise to show that for $\lambda > 1$ the solution $u_\lambda(x,t)$ blows up in finite time.

Now suppose that there are two solutions, $0 > w_2 \geq w_1$ of (5.6). By Theorem (5.2.5) the two solutions cannot have a point of tangency; hence $w_2 > w_1$ in Ω. Let

$$\lambda^* = \inf\{\lambda \,:\, \lambda w_2 > w_1\}.$$

Then $\lambda^* > 1$, hence $\lambda^* w_2$ is an upper solution; while $\lambda^* w_2$ and w_1 must have a point of tangency somewhere in $\bar{\Omega}$. Thus Theorem (5.2.5) is violated, and there cannot be two solutions of (5.6), one lying entirely on one side of the other.

Now consider the nonlinear problem

$$(\Delta + \mu)u - u^3 = 0, \qquad Bu = 0, \tag{5.7}$$

where μ is a parameter. We begin by showing that for $\mu < \mu_1$ the trivial solution is a stable equilibrium for the nonlinear parabolic equation

$$(\Delta + \mu)u - u^3 = u_t, \qquad Bu = 0.$$

As a comparison function for this problem we use the principal eigenfunction of the Laplacian:

$$\Delta\psi_1 + \mu_1\psi_1 = 0, \qquad B\psi_1 = 0.$$

We have

$$(\Delta + \mu)\lambda\psi_1 - (\lambda\psi_1)^3 = \lambda\psi_1(\mu - \mu_1 - \lambda^2\psi_1^2), \qquad B\psi_1 = 0. \tag{5.8}$$

By the Krein-Rutman theorem, $\psi_1 > 0$ in Ω. For $\mu < \mu_1$ it follows that $\lambda\psi_1$ is a lower solution for $\lambda < 0$ and an upper solution for $\lambda > 0$. Hence the origin (i.e. $u = 0$) is stable. These inequalities are reversed when $\mu > \mu_1$, hence the origin becomes unstable as μ crosses μ_1.

For $\mu < \mu_1$ the only solution of (5.7) is the trivial solution. We use the fact that the eigenvalues of the Laplacian decrase monotonically as the domain increases. Let $\Omega \subset \Omega'$ and let ψ_1', μ_1' be the principal eigenfunction and eigenvalue of the Laplacian on Ω'. Then $\psi_1' > 0$ on $\bar{\Omega}$, and $\mu_1' < \mu_1$. If u is any non-trivial solution of (5.7), put $u = w\psi_1'$ and substitute it into the partial differential equation to get

$$\psi_1'\Delta w + 2\nabla\psi_1' \cdot \nabla w + w(\Delta + \mu)\psi_1' - w^3\psi_1^3 = 0.$$

Since $\psi_1' > 0$ on $\bar{\Omega}$ we can divide through by this function to get

$$\Delta w + \frac{2}{\psi_1'}\nabla\psi_1' \cdot \nabla w + w(\mu - \mu_1' - w^2\psi_1'^2) = 0.$$

Since $\mu < \mu_1$ we choose Ω' so that $\mu < \mu_1' < \mu_1$; then the coefficient of w in the above equation is negative, and by the maximum principle, w and hence u vanishes identically.

We now show that a pair of solutions $\pm w$, $w > 0$, bifurcates from the trivial solution as μ crosses μ_1. From (5.8) we see that $\lambda\psi_1$ is a lower solution when $\mu > \mu_1$ and $0 < \lambda < \delta$ for some $\delta > 0$. Applying the same argument to ψ_1', we get

$$(\Delta + \mu)\lambda\psi_1' - (\lambda\psi_1')^3 = \lambda\psi_1'(\mu - \mu_1 - (\lambda\psi_1')^2).$$

Since $\psi_1' > 0$ on $\partial\Omega$ and since $\partial\Omega$ is compact, we can find a $\delta' > 0$ such that $\psi_1' \geq \delta' > 0$ on $\partial\Omega$. Therefore, for λ sufficiently large, $\lambda\psi_1'$ is an upper solution. The existence of a positive solution w of (5.7) then follows.

5.3 Crash Course in Elliptic Equations

In this section we summarize some of the principle facts about second order elliptic and parabolic boundary value problems which will be needed.

5.3.1 Regularity theory for elliptic equations

Given a function u defined in a domain $\Omega \subset \mathbb{R}^n$ the Hölder norm $||u||_\alpha$ is defined by

$$||u||_\alpha = \sup_\Omega |u| + H_\alpha(u),$$

where

$$H_\alpha(u) = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha}, \qquad 0 < \alpha \leq 1.$$

A function u is said to be Hölder continuous with exponent α in Ω if $H_\alpha(u) < +\infty$. The class of Hölder continuous functions on Ω is denoted by $C^\alpha(\Omega)$. The class of functions which are Hölder continuous along with their derivatives up to order k is denoted by $C^{k,\alpha}(\Omega)$. The associated norm is defined by

$$||u||_{k,\alpha} = \sum_{\beta\leq k} ||D^\beta u||_\alpha.$$

The spaces $C^{k,\alpha}(\Omega)$ form a *Banach algebra* under the norms $||\cdot||_{k,\alpha}$. That is, they form a Banach space, and in addition,

$$||uv||_{k,\alpha} \leq ||u||_{k,\alpha}||v||_{k,\alpha}.$$

Moreover, if $\alpha' \leq \alpha$ then $C^{k,\alpha}$ is compactly embedded in $C^{k,\alpha'}$; that is, bounded subset in $C^{k,\alpha}$ are compact in $C^{k,\alpha'}$.

We have previously introduced the Sobolev spaces $W^{k,p}$. A general reference is Adams, [1]. The following result is the basic Sobolev embedding lemma:

Theorem 5.3.1 *If Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, and $n < p < \infty$, then*

$$||u||_{1-n/p} \leq C(n,p,\Omega)||u||_{1,p}.$$

In the following, we take L to be the uniformly elliptic second order operator defined in §(5.1), and assume that the coefficients of L belong to $C^\alpha(\Omega)$ and that $\partial\Omega$ is of class $C^{2,\alpha}$. We assume g is defined on $\partial\Omega$ and has a $C^{2,\alpha}$ extension into the interior of Ω. We denote that extension also by g.

Theorem 5.3.2 *The linear elliptic boundary value problem*

$$Lu = h, \qquad Bu = g$$

has a unique smooth solution in class $C^{2,\alpha}$; moreover u satisfies the a priori estimate

$$||u||_{2,\alpha} \leq C(||f||_\alpha + ||g||_{2,\alpha}).$$

The constant C depends only on Ω, n, and α.

The following L_p estimates are also valid:

$$||u||_{2,p} \leq C(||f||_p + ||g||_{2,p}).$$

See Agmon, Douglis, and Nirenberg [2], especially theorems 7.3 and 15.2. The same estimates hold for operators $L + c(x)$, where $c \in C^\alpha$ and $L + c$ has a trivial kernel, i.e. $(L + c)u = 0$ implies $u = 0$.

In (5.3), (5.4) we may reduce the problem to the case $g = 0$, and we do that in the following discussion. Then L is a bounded linear operator from $C_0^{2,\alpha}(\Omega)$ to $C_0^\alpha(\Omega)$, or from $W^{2,p}$ to L^p. According to the theorem, its inverse, which we denote by G, is a continuous linear transformation from C^α to $C^{2,\alpha}$ or from L^p to $W^{2,p}$. Though we shall not prove it here, G is an integral operator whose kernel is the Green's function for L.

To show convergence of the sequence of upper solutions, write the sequence in the form

$$u^{n+1} = -G_A f_A(x, u^n), \qquad G_A = (L - A)^{-1}.$$

Since $u^n \downarrow U(x)$ and u^n is bounded, the sequence $f_A(x, u^n)$ converges in $L^p(\Omega)$ for any p. Since G_A is a continuous mapping from L^p to $W_0^{2,p}(\Omega)$, the solutions u^n converge in $W^{2,p}$. For $p > n$ the upper solutions are in $C^{1-n/p}$, by the Sobolev embedding theorem. Since G_A maps C^α boundedly into $C^{2,\alpha}$, the upper solutions are in $C^{2,\alpha}$ for any $0 < \alpha < 1$. Since the upper solutions converge pointwise and are uniformly bounded in $C^{2,\alpha}$ their limit $\bar{U}(x) \in C^{2,\alpha}$.

Let $u \in C^{2,\alpha}$ satisfy the nonlinear boundary value problem and write

$$u = -Gf(x,u).$$

If f is a C^∞ function of (x,u), and if the coefficients of L are C^∞, then by induction we may conclude that the solution u is C^∞. Likewise, if f is analytic in both variables, and the coefficients of L are analytic in the interior, then it follows that any bounded solution u is *a priori* analytic. This induction argument is commonly called a *bootstrap* argument.

A weak solution of the nonlinear boundary value problem is a function $u \in L^1_{loc}(\Omega)$ such that

$$(u, L^*\varphi) + (f(x,u),\varphi) = 0 \qquad \forall\ \varphi \in W^{2,p}(\Omega). \tag{5.9}$$

Theorem 5.3.3 *Let $u \in L^\infty(\Omega)$ satisfy* (5.9)*, where f is Hölder continuous in (x,u). Then $u \in C^{2,\alpha}(\Omega)$, and u satisfies the nonlinear boundary value problem* (5.3) (5.4)*.*

Proof: Put $w = -Gf(x,u)$. Then $w \in W^{2,p}_0(\Omega)$ for any $p > 1$, and

$$(w, L^*\varphi) = -(Gf, L^*\varphi) = -(f, G^*L^*\varphi) = -(f,\varphi) \qquad \forall \varphi \in C^2.$$

Hence $(u-w, L^*\varphi) = 0$ for all $\varphi \in C^2$. Now take $\varphi = G^*(u-w)$. Then $\varphi \in W^{2,p}$ and we get

$$(u-w, L^*G^*(u-w) = |u-w|_2^2 = 0.$$

Hence $u = w$ *a.e.* and so u may be modified on a set of measure zero so that $u \in W^{2,p}_0(\Omega)$. By the Sobolev embedding theorem u is Hölder continuous, and we may then repeat the bootstrap argument above. ■

5.3.2 The Fredholm Alternative

The Fredholm alternative for elliptic operators is fundamental to nonlinear analysis, especially in perturbation or bifurcation theory. We state it here. Let $f(x),\ c(x) \in C^\alpha(\Omega)$ and consider the elliptic boundary value problem

$$(L+c)u = f, \qquad Bu = 0. \tag{5.10}$$

If $c \le 0$ in Ω then by the maximum principle the homogeneous equation

$$(L+c)u = 0, \qquad Bu = 0 \tag{5.11}$$

has only the trivial solution. If c assumes positive values, the linear homogeneous equation may have one or more non-trivial solutions $\varphi + 1, \dots \varphi_n$. If so, then the adjoint equation

$$L^*u + cu = 0 \qquad Bu = 0 \tag{5.12}$$

also has n solutions $\varphi_1^a st, \dots \varphi_n^*$, and by the regularity theory, these lie in $C^{2,\alpha}$.

If the homogeneous equation reffellhom has only the trivial solution, then (5.10) has a unique solution $u \in C^{2,\alpha}$. If the homogeneous equation has n independent solutions then the adjoint equation also has n independent solutions $\varphi_j^* \in C^{2,\alpha}$. In that case (5.10) has a solution u if and only if

$$\int_\Omega f(x)\varphi_j^*\,dx = 0, \qquad j = 1, \dots n.$$

One way to prove the Fredholm alternative is the following. Convert (5.10) to an integral equation by applying the Green's operator for L:

$$u + Gcu = Gf.$$

By the *a priori* regularity results for elliptic operators, G is a bounded operator from C^α to $C^{2,\alpha}$. It follows that $Gf \in C^{2,\alpha}$, and the integral equation above holds in $C^{2,\alpha}$.

A compact (or completely continuous) operator G on a Banach space $\mathcal{E}$ is one that maps bounded sets into compact sets: that is, if $\{u_j\}$ is a bounded sequence in $\mathcal{E}$, then $\{Gu_j\}$ contains a convergent subsequence. By the elliptic regularity theory, $T = Gc$ is a compact operator, since it maps C^α continuously into $C^{2,\alpha}$ and since $C^{2,\alpha}$ is compactly embedded in C^α. The Fredholm alternative holds for functional equations of the form

$$(I + T)u = f$$

on a Banach space $\mathcal{E}$, where T is a compact operator. See Riesz and Nagy [16] for the theory when $\mathcal{E}$ is a Hilbert space, Dunford and Schwartz, volume I [7] when $\mathcal{E}$ is a Banach space.

5.3.3 The Krein-Rutman Theorem

Consider the eigenvalue problem

$$Lu + c(x)u + \lambda u = 0, \qquad Bu = 0.$$

For now, we us assume that $c(x) \leq 0$; hence 0 is not an eigenvalue of this equation, and $L + c$ is invertible. Denoting $-(L + c)^{-1}$ by G, we may rewrite the eigenvalue problem as an integral equation

$$u = \lambda G u,$$

where G is a compact integral operator on the Banach space $C^{2,\alpha}$.

A *cone* in a Banach space $\mathcal{E}$ is a closed subset K with the properties that i) $u,\ v \in K$ imply that $\alpha u + \beta v \in K$ for all $\alpha,\ \beta \geq 0$;

ii) $u,\ v \in K$ and $u \neq 0$ imply that $u + v \neq 0$. The set of non-negative functions on Ω forms a cone, which we denote by K, in $C^{2,\alpha}$, or in $W^{k,p}(\Omega)$ for $k \geq 0$ and $p \geq 1$. The interior of K consists of strictly positive functions.

An operator G is said to be *strongly positive* relative to K is for each $u \in \partial K$ there is an integer $n = n(u)$ such that $K^n u$ belongs to the interior of K. We leave it as an exercise to show, using the strong maximum principle, that $-(L + c)^{-1}$ is strongly positive with $n = 1$ for all u.

Theorem 5.3.4 [Krein-Rutman] *Let K be a cone in a Banach space $\mathcal{E}$, and let G be strongly positive with respect to K. Then G has one and only one eigenfunction φ in the interior of K, and the corresponding eigenvalue is real and simple.*

Corollary 5.3.5 *Let L be a uniformly elliptic operator on a domain Ω, with Hölder continuous coefficients. Then the eigenvalue with least real part is real and simple, and the corresponding eigenfunction is strictly positive on Ω.*

5.4 The method of moving planes

The method of moving planes was invented by Alexandrov to prove that global solutions of the nonlinear elliptic equation of constant mean curvature were spheres. The method was in turn extended and developed by Serrin, and later by Gidas, Ni, and Nirenberg to prove rotational symmetry of solutions of solutions of the semi-linear equation

$$\Delta u + f(u) = 0$$

on $\mathbb{R}^n$.

Here is an example of such a result; the proof presented here is due to J. Serrin and Henzhei Zou.

Theorem 5.4.1 *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of*

$$\Delta u + f(u) = 0, \qquad x \in \mathbb{R}^n; \qquad u(x) \to \infty \quad as \quad |x| \to \infty. \tag{5.13}$$

Suppose that f is locally Lipshitz continuous on $(0, \infty)$ and is non-increasing on $(0, \delta)$ for some $\delta > 0$.

Then u is radially symmetric about some point $p \in \mathbb{R}^n$; and, moreover,

$$\frac{\partial u}{\partial r} < 0, \qquad r > 0.$$

Proof: For γ real and fixed, define

$$\Sigma_\gamma = \{x_1 < \gamma\}, \qquad \Gamma = \{\, x_1 = \gamma \,\};$$

$$x^\gamma = (2\gamma - x_1, x_2, \ldots x_n).$$

We define $u^\gamma(x) = u(x^\gamma)$, the solution reflected in the plane Γ. Since the Laplacian is invariant under reflections, u^γ is also a solution of (5.13). Let $w = w^\gamma = u - u^\gamma$. Then

$$\Delta w + c_\gamma(x) w = 0, \qquad c_\gamma(x) = \frac{f(u) - f(u^\gamma)}{u - u^\gamma}.$$

Since f is Lipshitz, c is bounded on bounded sets in $\mathbb{R}^n$. Moreover, $c_\gamma \le 0$ when $0 < u$, $u^\gamma < \delta$, since f is non-increasing on this interval.

The proof proceeds in several steps.

Step 1. There exists γ_0 such that for all $\gamma \ge \gamma_0$, $w(x) \ge 0$ for all $x \in \Sigma_\gamma$.

Since $u \to 0$ as $|x| \to \infty$, there exists a γ_0 such that $0 < u(x) < \delta$ for $x_1 > \gamma_0$. Suppose $w < 0$ somewhere in Σ_γ. Since $w \to 0$ as $|x| \to \infty$ and $w = 0$ on Γ_γ, w must have a negative minimum at some point $y \in \Sigma_\gamma$. Then

$$u(y) - u^\gamma(y) = w(y) < 0;$$

hence $0 < u(y) < u^\gamma(y) < \delta$, since $y_1^\gamma > \gamma_0$. Therefore $c_\gamma(y) \le 0$, and in fact, $c_\gamma(x) \le 0$ for all x in a neighborhood of y. By Corollary (5.1.5), w is identically equal to a negative constant in a neighborhood of y, in contradiction to the fact that $w = 0$ on Γ_γ. *Step 2. Suppose* $w \geq 0$ *in* Σ_γ *for some* γ. *Then either* $w \equiv 0$ *in* Σ_γ *or*

$$w > 0 \text{ in } \Sigma_\gamma; \quad \text{and} \quad \frac{\partial w}{\partial x_1} < 0 \text{ on } \Gamma_\gamma.$$

This result is a consequence of Theorem (5.1.6). *Step 3. There exists a maximal closed subinterval* $[\mu, \infty)$ *such that* $w \geq 0$ *in* Σ_γ *for all* $\gamma \in [\mu, \infty)$.

It suffices to show there exists a γ such that $w^\gamma(x) < 0$ for some $x \in \Sigma_\gamma$. Let $z \in \mathbb{R}^n$ be fixed and choose $\gamma < 0$ so large that $\gamma < z_1$ and $0 < u(x) < u(z)$ for all $x \in \Sigma_\gamma$. Then $z^\gamma \in \Sigma_\gamma$, and

$$w(z^\gamma) = u(z^\gamma) - u(z) < 0.$$

Step 4. Let $[\mu, \infty)$ *be the maximal closed subinterval; then* $w^\mu \equiv 0$ *in* Σ_μ; *i.e., u is symmetric about the plane* Γ_μ.

If not, then by Step 2

$$w > 0 \quad \text{on} \quad \Sigma_\mu; \qquad \frac{\partial w}{\partial x_1} < 0 \qquad \text{on} \quad \Gamma_\mu. \tag{5.14}$$

On the other hand, since $[\mu, \infty)$ is maximal, there is a sequence $\{\gamma^k\}$ and points $x_k \in \Sigma_{\gamma_k}$ such that

$$w_k(\gamma_k) < 0, \qquad w_k = u(x) - u(x^{\gamma_k}).$$

Without loss of generality we may choose x_k to be the minimum of w_k in Σ_{γ_k}.

There are three possibilities: i) $|x_k| \to \infty$; ii) $x_k \to y \in \Sigma_\mu$; iii) $x_k \to y \in \Gamma_\mu$. In the first case we find a contradiction as in Step 1. In the second case, by continuity, it follows that $w_k(y) = 0$, which contradicts our assumption that $w > 0$ in Σ_μ. Finally, in case iii), there is another sequence $\{z_k\}$ such that $z_k \to y \in \Gamma_\mu$ and

$$\frac{\partial w_k}{\partial x_1}(z_k) \geq 0.$$

Again, by continuity,

$$\frac{\partial w}{\partial x_1}(y) \geq 0,$$

which contradicts the second inequality in (5.14).

Thus u is symmetric about Γ_μ. *Step 5.* We have now shown that in every direction there is a hyperplane Σ_μ about which u is symmetric. Choose n orthogonal hyperplanes with a common point of intersection $p \in \mathbb{R}^n$. Clearly $\nabla u(p) = 0$; and moreover, u attains a maximum at p. Furthermore, any other hyperplane of symmetry must also pass through p, since for a given direction, there is a unique maximal μ for which u is symmetric about Γ_μ. Thus u is symmetric about every hyperplane passing through p; and therefore u is radially symmetric about p. ■

5.5 Exercises

1. Prove a uniqueness theorem for the linear boundary value problem

$$(L+h)u = 0, \qquad Bu = 0$$

under the assumption that the principal eigenvalue $\lambda_1 > 0$. 2. Prove that the solution to the initial value problem

$$u_t = \Delta u - u^2, \qquad Bu = 0, \qquad u(x,0) < w(x)$$

blows up in finite time, where w is the non-trivial solution to the time independent problem. 3. Use the maximum principle to compare the principal eigenvalues of the Laplacian on two domains $\Omega \subset \Omega'$. 4. Prove the principle of linearized stability for solutions of (5.3) (5.4). That is, a stationary solution w is a stable equilibrium for the associated parabolic equation if and only if the principal eigenvalue of the linearized operator

$$(L + f_u(x,w) + \lambda)\psi = 0,, \qquad B\psi = 0$$

is positive.

5. The nonlinear problem

$$\Delta T + \lambda e^{-E/RT} = 0, \qquad T\Big|_{\partial\Omega} = T_0$$

arises in combustion theory. Prove that for some values of λ the problem has three solutions. 6. The equation

$$F(x, u, Du, D^2u) = 0,$$

where Du denotes the first order derivatives of u and D^2u the second order derivatives, is said to be elliptic with respect to a function $u \in C^2(\Omega)$ if

$$\sum_{j,k=1}^{n} \frac{\partial F}{\partial u_{jk}} \xi_j \xi_k > 0, \qquad x \in \Omega.$$

Formulate and prove an extension of Theorem (5.2.5) to such a fully nonlinear elliptic equation.

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