BEGINNING PARTIAL DIFFERENTIAL EQUATIONS

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A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York • Chichester • Weinheim • Brisbane • Singapore • Toronto

This book is printed on acid-free paper. ∞

Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

O'Neil, Peter V.
Beginning partial differential equations / Peter V. O'Neil.
p. cm.
"A Wiley-Interscience publication."
Includes index.
ISBN 0-471-23887-2 (cloth : alk. paper)
1. Differential equations. Partial. I. Title.
QA377.054 1999 98-44621
515'.353—dc21

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

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PREFACE

This book is intended as a first course in partial differential equations. Topics include characteristics, canonical forms, well-posed problems, and properties of solutions, as well as techniques for writing expressions for solutions.

Our objective is to provide an introduction to this important field of mathematics, as well as an entry point, for those who wish it, to the modern, more abstract elements of partial differential equations.

Because this is an introductory treatment, we attempt a balance between theory and technique. Computational facility and theoretical understanding reinforce each other, and both are important for later work in related areas, such as mathematical physics, differential geometry, or analytic number theory. Although the emphasis is on the mathematics, we also point out physical interpretations, for those circumstances in which an initial-boundary value problem models a physical setting.

The exercises have two purposes. Some are computational, ranging from routine to challenging. Answers for many of these are included at the end of the book. Other exercises provide additional information about partial differential equations, or extensions of the material of the text. Many of these come with hints for the development of a proof or a derivation.

Partial differential equations invite graphical representation and experimentation. Sometimes we can visualize a solution as a surface. Further, some partial differential equations model physical phenomena, and it is interesting and instructive to couple mathematics with physical intuition by observing the evolution with time of a solution that represents heat conduction or wave motion. We can also experiment with the influence of different parameters on the physical phenomenon represented by the partial differential equation, such as the effect of the specific heat of a metal bar on the way it conducts heat energy, or the way the tension and density of a guitar string influence the way it vibrates. Parts of some exercises pursue such issues, and require the use of a computational package, such as MAPLE or Mathematica. Readers who do not have access to suitable hardware and software can skip over these exercises.

Preliminary versions of this book have been tested at the University of Alabama at Birmingham over the past five years.

PREREQUISITES

The reader should be familiar with standard properties of real-valued functions of *n* real variables, vector calculus (theorems of Green and Gauss), topics from the standard post-calculus course in elementary ordinary differential equations, and convergence of series and improper integrals. Occasional mention is made of uniform convergence. Topics from Fourier analysis are not assumed as background and are included. Several discussions (the use of conformal mappings to solve the Dirichlet problem, contour integration to evaluate certain integrals, and Euler's formula) use complex numbers and complex function theory.

Access to a computational program, such as MAPLE, is useful both in performing calculations and in studying properties of solutions, convergence of Fourier series, and other issues of interest in partial differential equations. Such access is not a prerequisite to reading this book, but some exercises inviting computer study and experimentation have been included for those who have it.

ACKNOWLEDGMENTS

I would like to thank my colleagues at UAB for helpful conversations and suggestions, several classes of students for tolerating the use of preliminary versions of this book in the form of course notes, and the editorial staff of John Wiley & Sons for their professionalism in bringing the project to completion.

1

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

1.1 PRELIMINARY NOTATION AND CONCEPTS

A *partial differential equation* is an equation that contains at least one partial derivative. For example,

$$\frac{\partial u}{\partial x} - x\,\frac{\partial u}{\partial y} = xuy^2,$$

$$\frac{\partial^4 w}{\partial x^4} - 3\,\frac{\partial^2 w}{\partial z \partial u} + \frac{\partial^3 w}{\partial u^3} + \cos(xu)\,\frac{\partial^3 w}{\partial x \partial y \partial z} - xyzu\,\frac{\partial w}{\partial x} = 0$$

and

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = f(x, y, z)$$

are partial differential equations. Such equations are of interest in mathematics (for example, in the study of curvature and surfaces) and in modeling phenomena in the sciences, engineering, economics, ecology, and other areas. For the moment we will be concerned with developing the vocabulary and notation with which to engage in a discussion of partial differential equations.

There is some economy in using subscripts to denote partial derivatives. In this notion, $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xy} = \partial^2 u/\partial y \partial x$, and so on. The partial differential equations listed above can be written, respectively,

$$u_x - xu_y = xuy^2,$$

$$w_{xxxx} - 3w_{uz} + w_{uuu} + \cos(xu)w_{zyx} - xyzuw_x = 0$$

and

$$h_{xx} + h_{yy} + h_{zz} = f(x, y, z).$$

A *solution* of a partial differential equation is any function that satisfies the equation. Often we seek a solution satisfying certain conditions, and for the independent variables confined to a specified set of values.

As an example of a solution, the equation

$$4u_x + 3u_y + u = 0 \tag{1.1}$$

has solution

$$u(x, y) = e^{-x/4}f(3x - 4y)$$

in which f can be any differentiable function of a single variable. We will verify this by substituting $u(x, y)$ into the partial differential equation. Chain rule differentiations yield

$$\begin{aligned} u_x &= -\frac{1}{4}e^{-x/4}f(3x - 4y) + e^{-x/4}\frac{d}{d(3x - 4y)}[f(3x - 4y)]\frac{d(3x - 4y)}{dx} \\ &= -\frac{1}{4}e^{-x/4}f(3x - 4y) + 3e^{-x/4}f'(3x - 4y) \end{aligned}$$

and

$$u_y = -4e^{-x/4}f'(3x - 4y).$$

Upon substitution into equation 1.1 we obtain

$$\begin{aligned} 4u_x + 3u_y + u = -e^{-x/4}f(3x - 4y) &+ 12e^{-x/4}f'(3x - 4y) \\ &- 12e^{-x/4}f'(3x - 4y) + e^{-x/4}f(3x - 4y) = 0 \end{aligned}$$

for all (x, y).

As another example, consider the equation

$$u_{xx} - 9u_{yy} = 0. \tag{1.2}$$

It is routine to check that

$$u(x, y) = f(3x + y) + g(3x - y)$$

is a solution for any twice differentiable functions f and g of a single variable. For example, we could choose $f(t) = \sin(t)$ and $g(t) = e^{-t} + t^2 - \cos(t)$ to obtain the solution

$$u(x, y) = \sin(3x + y) + e^{-3x+y} + (3x - y)^2 - \cos(3x - y).$$

In view of the latitude in choosing f and g, equation 1.2 has infinitely many solutions (as does equation 1.1).

The *order* of a partial differential equation is the order of the highest partial derivative occurring in the equation. Equation 1.1 is of order one and equation 1.2 is of order two.

A partial differential equation is *linear* if it is linear in the unknown function and its partial derivatives. An equation that is not linear is *nonlinear*. For example,

$$x^2 u_{xx} - y u_{xy} = u$$

is a linear partial differential equation, while

$$x^2 u_{xx} - y u_{xy} = u^2$$

is nonlinear because of the u^2 term, and

$$(u_{xx})^{1/2} - 4u_{yy} = xu$$

is nonlinear because of the $(u_{xx})^{1/2}$ term.

A partial differential equation is *quasi-linear* if it is linear in its highest order derivative terms. For example, the second order equation

$$u_{xx} + 4y u_{yy} - (u_x)^3 + u_x u_y = \cos(u)$$

is quasi-linear, being linear in its second derivative terms u_{xx} and u_{yy}. This equation is nonlinear because of the $\cos(u)$, $u_x u_y$ and $(u_x)^3$ terms. Of course, any linear equation is also quasi-linear.

We are now ready to begin studying first order partial differential equations.

EXERCISE 1 Show that

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution of $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x, y, z) \neq (0, 0, 0)$.

EXERCISE 2 Show that $u(x, t) = f(x + ct) + g(x - ct)$ is a solution of

$$u_{tt} = c^2 u_{xx}$$

for any twice differentiable functions f and g of one variable; c is a positive constant.

EXERCISE 3 Show that

$$u(x, t) = \frac{1}{2}(\varphi(x + ct) + \varphi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

is a solution of $u_{tt} = c^2 u_{xx}$ for any φ that is twice differentiable and ψ that is differentiable for all real x. c is a positive constant. Show that this solution satisfies the conditions

$$u(x, 0) = \varphi(x),\ u_t(x, 0) = \psi(x)$$

for all real x.

EXERCISE 4 Show that, if p is a continuously differentiable function of one variable, then the first order partial differential equation

$$u_t = p(u)u_x$$

has solution implicitly defined by

$$u(x, t) = \varphi(x + p(u)t),$$

in which φ can be any continuously differentiable function of one variable.

Use this idea to determine (perhaps implicitly) a solution of each of the following equations:

1. $u_t = ku_x$, with k a nonzero constant
2. $u_t = uu_x$
3. $u_t = \cos(u)u_x$
4. $u_t = e^u u_x$
5. $u_t = u \sin(u)u_x$

EXERCISE 5 Show that

$$u(x, y) = \ln((x - x_0)^2 + (y - y_0)^2)$$

satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ for all pairs (x, y) of real numbers except (x_0, y_0).

EXERCISE 6 Let v and w be solutions of

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + g(x, y)u = 0.$$

Show that $\alpha w + \beta v$ is also a solution for any numbers α and β.

EXERCISE 7 In each of the following, classify the equation as linear, nonlinear but quasi-linear, or not quasi-linear.

1. $u^2 u_{xx} + u_y = \cos(u)$
2. $x^2 u_x + y^2 u_y + u_{xy} = 2xy$
3. $(x - y)u_x^2 + u_{xy} = 1$
4. $(x - y)u_x^2 + 2u_y = 4y$
5. $x^2 u_{yy} - yu_{xx} = \tan(u)$
6. $u_x + u_y^2 - u_{xx} = 4$
7. $u_x - u_x u_y - u_y = 0$
8. $uu_x + u_{xy} = u^2$
9. $u_{xy} - u_x^2 + u_y^2 - \sin(u_x) = 0$
10. $u_y/u_x = x^2$

EXERCISE 8 Let k be a positive constant. Let

$$u(x, t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4kt} f(\xi)\, d\xi,$$

in which f is continuous on the real line. Show that $u_t = ku_{xx}$ for $-\infty < x < \infty$, $t > 0$.

Determine $u(x, t)$ when $f(x) \equiv 1$. *Hint*: Use a change of variables and the standard result that

$$\int_{-\infty}^{\infty} e^{-w^2}\, dw = \sqrt{\pi}.$$

1.2 THE LINEAR EQUATION

Consider the linear first order partial differential equation in two independent variables:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y). \tag{1.3}$$

We assume that a, b, c, and f are continuous in some region of the plane, and that $a(x, y)$ and $b(x, y)$ are not both zero for the same (x, y).

We will show how to solve equation 1.3. The key is to determine a change of variables

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y)$$

which transforms equation 1.3 to the simpler linear equation

$$w_\xi + h(\xi, \eta)w = F(\xi, \eta) \tag{1.4}$$

where $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. We will define this transformation in such a way that it is one-to-one, at least for all (x, y) in some set $\mathcal{D}$ of points in the x, y-plane. On $\mathcal{D}$, then, we can, at least in theory, solve for x and y as functions of ξ and η. To insure this we will require that the Jacobian of the transformation does not vanish in $\mathcal{D}$:

$$J = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

for (x, y) in $\mathcal{D}$.

Begin the search for a suitable transformation by computing chain rule derivatives:

$$u_x = w_\xi\xi_x + w_\eta\eta_x$$

and

$$u_y = w_\xi\xi_y + w_\eta\eta_y.$$

Substitute these into equation 1.3 to obtain:

$$a(w_\xi\xi_x + w_\eta\eta_x) + b(w_\xi\xi_y + w_\eta\eta_y) + cw = f.$$

Write this equation as

$$(a\xi_x + b\xi_y)w_\xi + (a\eta_x + b\eta_y)w_\eta + cw = f. \tag{1.5}$$

This is nearly in the form of equation 1.4 if we choose $\eta = \psi(x, y)$ so that

$$a\eta_x + b\eta_y = 0$$

for (x, y) in $\mathcal{D}$. If $\eta_y \neq 0$ this requires that

$$\frac{\eta_x}{\eta_y} = -\frac{b}{a}.$$

Suppose for the moment that there is such an η. Putting $\eta(x, y) = c$, with c an arbitrary constant, then

$$d\eta = \eta_x\, dx + \eta_y\, dy = 0$$

implies that

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{b}{a}.$$

This means that $\eta = \psi(x, y)$ is an integral of the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \tag{1.6}$$

Equation 1.6 is called the *characteristic equation* of the linear equation 1.3. The equation $\eta(x, y) = c$ defines a family of curves in the plane called *characteristic curves*, or *characteristics*, of equation 1.3. We will say more about these in the next section.

Thus far we have found that we can make the coefficient of w_η in the transformed equation 1.5 vanish if we choose $\eta = \psi(x, y)$, with $\psi(x, y) = c$ an equation defining the general solution of the characteristic equation 1.6. With this step alone equation 1.5 comes very close to the transformed equation 1.4 we want to achieve. We can now choose ξ to suit our convenience and the condition that $J \neq 0$. One simple choice is

$$\xi = \varphi(x, y) = x.$$

With this choice,

$$J = \begin{vmatrix} 1 & 0 \\ \eta_x & \eta_y \end{vmatrix} = \eta_y$$

and this is nonzero in $\mathcal{D}$ by previous assumption.

Now from equation 1.5, the change of variables

$$\xi = x,\ \eta = \psi(x, y)$$

transforms equation 1.3 to

$$a(x, y)w_\xi + c(x, y)w = f(x, y).$$

To complete the transformation to the form of equation 1.4, first write $a(x, y)$, $c(x, y)$, and $f(x, y)$ in terms of ξ and η to obtain

$$A(\xi, \eta)w_\xi + C(\xi, \eta)w = p(\xi, \eta).$$

Finally, restricting the variables to a set in which $A(\xi, \eta) \neq 0$, we have

$$w_\xi + \frac{C}{A} w = \frac{p}{A}$$

and this is in the form of equation 1.4 with

$$h(\xi, \eta) = \frac{C(\xi, \eta)}{A(\xi, \eta)} \text{ and } F(\xi, \eta) = \frac{p(\xi, \eta)}{A(\xi, \eta)}.$$

Example 1 Consider the linear equation

$$x^2u_x + yu_y + xyu = 1.$$

This is equation 1.3 with $a(x, y) = x^2$, $b(x, y) = y$, $c(x, y) = xy$ and $f(x, y) = 1$. We will transform this equation to the simpler equation 1.4.

The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a} = \frac{y}{x^2}.$$

Write

$$\frac{1}{y}\, dy = \frac{1}{x^2}\, dx,$$

integrate and rearrange terms to obtain

$$\ln(y) + \frac{1}{x} = c$$

for $y > 0$ and $x \neq 0$. This is an integral of the characteristic equation and we choose

$$\eta = \psi(x, y) = \ln(y) + \frac{1}{x}.$$

Graphs of $\ln(y) + 1/x = c$ are the characteristics of this partial differential equation.

Upon choosing $\xi = x$ we have the Jacobian

$$J = \eta_y = \frac{1}{y} \neq 0$$

as required.

Since $\xi = x$,

$$\eta = \ln(y) + \frac{1}{\xi},$$

so

$$\ln(y) = \eta - \frac{1}{\xi}$$

and

$$y = e^{\eta - 1/\xi}.$$

Now apply the transformation

$$\xi = x,\ \eta = \ln(y) + \frac{1}{x},$$

with

$$w(\xi, \eta) = u(x, y).$$

Compute

$$u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta \left(-\frac{1}{x^2}\right) = w_\xi - \frac{1}{\xi^2} w_\eta$$

and

$$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\eta \frac{1}{y} = w_\eta \frac{1}{e^{\eta - 1/\xi}}.$$

The partial differential equation transforms to

$$\xi^2 \left(w_\xi - \frac{1}{\xi^2} w_\eta\right) + e^{\eta - 1/\xi} w_\eta \frac{1}{e^{\eta - 1/\xi}} + \xi e^{\eta - 1/\xi} w = 1,$$

or

$$\xi^2 w_\xi + \xi e^{\eta - 1/\xi} w = 1.$$

Then

$$w_\xi + \frac{1}{\xi} e^{\eta - 1/\xi} w = \frac{1}{\xi^2},$$

and this has the form of equation 1.4, in any region of the ξ, η-plane with $\xi \neq 0$. ■

The point to transforming equation 1.3 to the form of equation 1.4 is that we can solve this transformed equation. Think of

$$w_\xi + h(\xi, \eta) w = F(\xi, \eta)$$

as a linear first order ordinary differential equation in ξ, with η carried along as a parameter. Following the method for ordinary differential equations, multiply the differential equation by

$$e^{\int h(\xi,\eta) d\xi}$$

to obtain

$$e^{\int h(\xi,\eta) d\xi} w_\xi + h(\xi, \eta) e^{\int h(\xi,\eta) d\xi} w = F(\xi, \eta) e^{\int h(\xi,\eta) d\xi}.$$

Recognize this as

$$\frac{\partial}{\partial \xi} \left(e^{\int h(\xi,\eta) d\xi} w \right) = F(\xi, \eta) e^{\int h(\xi,\eta) d\xi}.$$

Integrate with respect to ξ. Since η is being carried through this process as a parameter, the constant of integration may depend on η. We obtain

$$e^{\int h(\xi,\eta) d\xi} w = \int F(\xi, \eta) e^{\int h(\xi,\eta) d\xi} \, d\xi + g(\eta),$$

in which g is any differentiable function of one variable. Then

$$w(\xi, \eta) = e^{-\int h(\xi,\eta) d\xi} \int F(\xi, \eta) e^{\int h(\xi,\eta) d\xi} \, d\xi + g(\eta) e^{-\int h(\xi,\eta) d\xi}. \tag{1.7}$$

This is the *general solution* of the transformed equation (by general solution, we mean one that contains an arbitrary function). Now obtain the general solution of the original equation 1.3 by substituting $\xi = \xi(x, y)$, $\eta = \eta(x, y)$. This general solution will have the form

$$u(x, y) = e^{\alpha(x,y)}[M(x, y) + g(\psi(x, y))]. \tag{1.8}$$

in which g is any differentiable function of one variable.

Example 2 Consider the constant coefficient equation

$$au_x + bu_y + cu = 0$$

in which a, b, and c are numbers. Assume that $a \neq 0$. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a}$$

with general solution defined by the equation

$$bx - ay = c.$$

Put

$$\xi = x, \ \eta = bx - ay.$$

The characteristics of this differential equation are the straight line graphs of $bx - ay = c$.

With this transformation, we find by a routine calculation that the partial differential equation transforms to

$$aw_\xi + cw = 0$$

or

$$w_\xi + \frac{c}{a} w = 0.$$

Multiply this equation by $e^{\int (c/a) d\xi}$, or $e^{c\xi/a}$, to get

$$e^{c\xi/a} w_\xi + \frac{c}{a} w e^{c\xi/a} = 0,$$

hence

$$\frac{\partial}{\partial \xi} (e^{c\xi/a} w) = 0.$$

Integrate with respect to ξ to get

$$e^{c\xi/a}w = g(\eta)$$

in which g can be any differentiable function of one variable. Then

$$w(\xi, \eta) = e^{-c\xi/a}g(\eta).$$

Finally, transform this solution back in terms of x and y:

$$u(x, y) = e^{-cx/a}g(bx - ay).$$

This solution is readily verified by substitution into the partial differential equation. ■

Observe that the solution in Example 2 has the form specified by equation 1.8.

Example 3 Consider

$$u_x + \cos(x)u_y + u = xy.$$

The characteristic equation is

$$\frac{dy}{dx} = \cos(x)$$

with general solution defined by

$$y - \sin(x) = c.$$

Put

$$\xi = x, \eta = y - \sin(x).$$

Graphs of $y - \sin(x) = c$ are the characteristics of this partial differential equation.

Now we have

$$y = \eta + \sin(x) = \eta + \sin(\xi)$$

and the partial differential equation transforms to

$$w_\xi + w = \xi[\eta + \sin(\xi)].$$

Multiply this equation by $e^{\int d\xi}$, which is e^{ξ}, to obtain

$$e^{\xi}w_{\xi} + we^{\xi} = \eta\xi e^{\xi} + \xi e^{\xi}\sin(\xi).$$

Write this equation as

$$\frac{\partial}{\partial\xi}(we^{\xi}) = \eta\xi e^{\xi} + \xi e^{\xi}\sin(\xi).$$

Integrate to obtain

$$\begin{aligned} we^{\xi} &= \int \eta\xi e^{\xi}\,d\xi + \int \xi e^{\xi}\sin(\xi)\,d\xi \\ &= \eta e^{\xi}(\xi - 1) + \frac{1}{2}\xi e^{\xi}(\sin(\xi) - \cos(\xi)) + \frac{1}{2}e^{\xi}\cos(\xi) + g(\eta). \end{aligned}$$

Then

$$w(\xi, \eta) = \eta(\xi - 1) + \frac{1}{2}\xi(\sin(\xi) - \cos(\xi)) + \frac{1}{2}\cos(\xi) + e^{-\xi}g(\eta).$$

Finally

$$u(x, y) = (y - \sin(x))(x - 1) + \frac{1}{2}x(\sin(x) - \cos(x)) + \frac{1}{2}\cos(x) + e^{-x}g(y - \sin(x)),$$

in which g is any differentiable function of a single variable. ■

Contrast the idea of the general solution for the linear first order ordinary differential equation with that for the linear first order partial differential equation. In the former case, the general solution of

$$y' + d(x)y = p(x)$$

contains an arbitrary constant. Graphs of the solutions obtained by making choices of the constant are curves in the (x, y) plane. If we require that $y(x_0) = y_0$, then we pick out the unique solution corresponding to the curve passing through (x_0, y_0).

By contrast, if u is the general solution of the linear first order partial differential equation 1.3, then $z = u(x, y)$ defines a family of surfaces in 3-space, each surface corresponding to a choice of the arbitrary function g in equation

1.8. In the next section we will investigate the kind of information that should be given in order to pick out one of these surfaces and determine a unique solution.

EXERCISE 9 For each of the following partial differential equations, (a) solve the characteristic equation and sketch graphs of some of the characteristics, (b) define a transformation of the partial differential equation to the form of equation 1.4 and obtain the transformed equation, (c) find the general solution of the transformed equation, (d) find the general solution of the given equation, and (e) verify the solution by substituting it into the partial differential equation.

1. $3u_x + 5u_y - xyu = 0$
2. $u_x - u_y + yu = 0$
3. $u_x + 4u_y - xu = x$
4. $-2u_x + u_y - yu = 0$
5. $xu_x - yu_y + u = x$
6. $x^2u_x - 2u_y - xu = x^2$
7. $u_x - xu_y = 4$
8. $x^2u_x + xyu_y + xu = x - y$
9. $u_x + u_y - u = y$
10. $u_x - y^2u_y - yu = 0$
11. $u_x + yu_y + xu = 0$
12. $xu_x + yu_y + 2 = 0$

EXERCISE 10 Find the general solution of

$$u_x + \alpha(y - 1)u_y = \frac{1}{2}\beta f(x)(y - 1)u$$

in which α and β are real numbers and f is continuous on the real line. Use the general solution to find the solution satisfying

$$u(0, y) = y^n,$$

in which n is a nonnegative integer.

1.3 THE SIGNIFICANCE OF CHARACTERISTICS

In the preceding section we mentioned characteristics but did not actually do anything with them. Now we will investigate their significance, beginning with an example that will be instructive for the point we want to make.

Consider

$$2u_x + 3u_y + 8u = 0.$$

The characteristic equation is

$$\frac{dy}{dx} = \frac{3}{2}$$

and the characteristics are the straight line graphs of $3x - 2y = c$.

Using the method of Section 1.2, we find that this partial differential equation has general solution

$$u(x, y) = e^{-4x}g(3x - 2y)$$

in which g can be any differentiable function defined over the real line.

Notice that simply specifying that the solution is to have a given value at a particular point does not uniquely determine g, and hence does not determine a unique solution, as occurs with ordinary differential equations.

Now suppose we specify values of $u(x, y)$ along a curve Γ in the plane. To be specific for this example, suppose we choose Γ as the x-axis and give values of $u(x, y)$ at points on Γ, say

$$u(x, 0) = \sin(x).$$

We need

$$u(x, 0) = e^{-4x}g(3x) = \sin(x)$$

so

$$g(3x) = e^{4x} \sin(x).$$

Putting $t = 3x$,

$$g(t) = e^{4t/3} \sin(t/3).$$

This determines g and the solution satisfying the condition $u(x, 0) = \sin(x)$ on Γ is

$$\begin{aligned} u(x, y) = e^{-4x}g(3x - 2y) &= e^{-4x}e^{4(3x-2y)/3} \sin\left(\frac{1}{3}(3x - 2y)\right) \\ &= e^{-8y/3} \sin\left(x - \frac{2}{3}y\right). \end{aligned}$$

In this example, specifying values of u along the x-axis uniquely determined the arbitrary function in the general solution, and hence determined the unique solution of the partial differential equation having these given values.

Next seek a solution having given values along the line $y = x$, say

$$u(x, x) = x^4.$$

From the general solution, this requires that

$$u(x, x) = e^{-4x}g(x) = x^4$$

and we can choose

$$g(x) = x^4 e^{4x}$$

to obtain the unique solution

$$u(x, y) = e^{-4x}g(3x - 2y) = e^{8(x-y)}(3x - 2y)^4$$

satisfying $u(x, x) = x^4$.

Despite these two successes, not every curve in the plane can be used to determine g. Suppose we choose Γ to be the line $3x - 2y = 1$, and prescribe values $u(x, y)$ is to have along Γ, say

$$u\left(x, \quad \frac{1}{2}(3x - 1)\right) = x^2.$$

Now we must choose g so that

$$e^{-4x}g\left(3x - 2\,\frac{1}{2}(3x - 1)\right) = x^2$$

and this requires that

$$g(1) = e^{4x}x^2.$$

This is impossible, hence there is no solution taking the value x^2 at points (x, y) on this line.

Why did some choices of Γ give a solution, and another choice no solution? The difference was that the x-axis and the line $y = x$ are not characteristics of the partial differential equation, while the line $3x - 2y = 1$ is a characteristic.

To understand the significance of characteristics in the context of existence and uniqueness of solutions, go back to the general solution 1.8 of the linear first order partial differential equation 1.3. This general solution is

$$u(x, y) = e^{\alpha(x,y)}[M(x, y) + g(\psi(x, y))].$$

Suppose we prescribe $u(x, y) = q(x)$ along a characteristic. Now a characteristic is specified by $\psi(x, y) = k$. If $y = y(x)$ along this characteristic, then

$$q(x) = e^{\alpha(x,y(x))}[M(x, y(x)) + g(k)]$$

or

$$q(x) = e^{\alpha(x,y(x))}[M(x, y(x)) + C], \tag{1.9}$$

in which C is constant. The functions $M(x, y)$ and $\alpha(x, y)$ are determined by the partial differential equation, and are not under our control, so equation 1.9 places a constraint on the given data function $q(x)$. If $q(x)$ is not of this form for any constant C, then there is no solution taking on these prescribed values on Γ. On the other hand, if $q(x)$ is of this form for some C, then there are infinitely many such solutions, because we can choose for g any differentiable function such that $g(k) = C$.

Example 4 Consider

$$xu_x + 2x^2u_y - u = x^2e^x. \tag{1.10}$$

First we will find the general solution. The characteristic equation is

$$\frac{dy}{dx} = 2x$$

and this has general solution defined by $y - x^2 = k$. The characteristics are parabolas. Let

$$\xi = x,\ \eta = y - x^2$$

to obtain

$$\xi w_\xi - w = \xi^2 e^\xi,$$

which we write as

$$w_\xi - \frac{1}{\xi} w = \xi e^\xi.$$

Multiply this equation by $e^{\int(-1/\xi)d\xi}$, which is $1/\xi$, to obtain

$$\frac{1}{\xi} w_{\xi} - \frac{1}{\xi^2} w = e^{\xi},$$

or

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\xi} w\right) = e^{\xi}.$$

Integrate with respect to ξ to get

$$\frac{1}{\xi} w = e^{\xi} + g(\eta)$$

so

$$w = \xi e^{\xi} + \xi g(\eta).$$

The general solution of equation 1.10 is

$$u(x, y) = xe^{x} + xg(y - x^2).$$

We will now attempt to find solutions satisfying given conditions along various curves.

Suppose first we seek a solution such that $u(x, y) = \sin(x)$ on the curve $y = x^2 + 4$. Notice that information is being specified along a characteristic. We will need

$$u(x, x^2 + 4) = xe^{x} + xg(4) = \sin(x).$$

We must be able to find a constant C such that

$$xe^{x} + Cx = \sin(x)$$

for all x, and this is impossible. There is no solution satisfying the requested condition.

Next suppose we want a solution such that $u(x, y) = xe^{x} - x$ on the parabola $y = x^2 + 4$. Now we need

$$u(x, x^2 + 4) = xe^{x} + xg(4) = xe^{x} - x.$$

This equation requires that $g(4) = -1$. This problem has infinitely many solutions because we can choose g to be any differentiable function of one variable such that $g(4) = -1$. Even though data is specified on a characteristic, the form of the data allows infinitely many solutions.

Finally, suppose we want a solution such that $u(x, y) = \cos(x)$ along the (noncharacteristic) parabola $y = x^2 + 4x$. Now we need

$$u(x, x^2 + 4x) = xe^x + xg(4x) = \cos(x).$$

This requires that

$$g(4x) = \frac{\cos(x) - xe^x}{x}.$$

Choose

$$g(t) = 4\,\frac{\cos(t/4) - \frac{t}{4}e^{t/4}}{t}$$

for, say, $t > 0$. The solution of the problem (for $x > 0$) is

$$\begin{aligned} u(x, y) &= xe^x + xg(y - x^2) \\ &= xe^x + 4x\left(\frac{\cos\left(\frac{y - x^2}{4}\right) - \frac{1}{4}(y - x^2)e^{(y-x^2)/4}}{y - x^2}\right). \end{aligned}$$ ■

The problem of finding a solution of equation 1.3 taking on prescribed values on a given curve is called a *Cauchy problem* (for the linear equation), and the given information on the curve is called *Cauchy data*. Our examples suggest that we can expect a unique solution of a Cauchy problem if the curve is not characteristic, and no solution or infinitely many solutions if the curve is characteristic.

EXERCISE 11 For each of the following partial differential equations, solve the characteristic equation and sketch graphs of some of the characteristics, find the general solution of the partial differential equation, and attempt to find solutions satisfying the Cauchy data on the given curves.

1. $3yu_x - 2xu_y = 0$
   (a) Find a solution satisfying $u(x, y) = x^2$ on the line $y = x$.
   (b) Find a solution satisfying $u(x, y) = 1 - x^2$ on the line $y = -x$.
   (c) Find a solution satisfying $u(x, y) = 2x$ on the ellipse $3y^2 + 2x^2 = 4$.

2. $u_x - 6u_y = y$
   (a) Find a solution satisfying $u(x, y) = e^x$ on the line $y = -6x + 2$.
   (b) Find a solution satisfying $u(x, y) = 1$ on the parabola $y = -x^2$.
   (c) Find a solution satisfying $u(x, y) = -4x$ on the line $y = -6x$.
3. $4u_x + 8u_y - u = 1$
   (a) Find a solution satisfying $u(x, y) = \cos(x)$ on the line $y = 3x$.
   (b) Find a solution satisfying $u(x, y) = x$ on the line $y = 2x$.
   (c) Find a solution satisfying $u(x, y) = 1 - x$ on the curve $y = x^2$.
4. $-4yu_x + u_y - yu = 0$
   (a) Find a solution satisfying $u(x, y) = x^3$ on the line $x + 2y = 3$.
   (b) Find a solution satisfying $u(x, y) = -y$ on $y^2 = x$.
   (c) Find a solution satisfying $u(x, y) = 2$ on $x + 2y^2 = 1$.
5. $yu_x + x^2u_y = xy$
   (a) Find a solution satisfying $u(x, y) = 4x$ on the curve $y = (1/3)x^{3/2}$.
   (b) Find a solution satisfying $u(x, y) = x^3$ on the curve $3y^2 = 2x^3$.
   (c) Find a solution satisfying $u(x, y) = \sin(x)$ on the line $y = 0$.
6. $y^2u_x + x^2u_y = y^2$
   (a) Find a solution satisfying $u(x, y) = x$ on $y = 4x$.
   (b) Find a solution satisfying $u(x, y) = -2y$ on $y^3 = x^3 - 2$.
   (c) Find a solution satisfying $u(x, y) = y^2$ on $y = -x$.

1.4 THE QUASI-LINEAR EQUATION AND THE METHOD OF CHARACTERISTICS

Consider the first order quasi-linear partial differential equation

$$f(x, y, u)u_x + g(x, y, u)u_y = h(x, y, u) \tag{1.11}$$

in which we seek a solution u as a function of the independent variables x and y. This is linear in the highest partial derivatives (which in the first order case are of order one), but may be nonlinear in u.

In the linear case we defined characteristics to be certain curves in the x, y-plane. Given a solution $u(x, y)$ of a linear equation and a characteristic Γ, we can define a curve on the surface $u = u(x, y)$ whose projection in the x, y-plane is Γ (Figure 1.1). In this context the curve in x, y, u-space is also called a characteristic, and its projection into the x, y-plane is called a *characteristic trace*.

FIGURE 1.1 A characteristic C on a solution surface, projecting onto a characteristic trace.

For the quasi-linear equation 1.11, characteristics are curves in x, y, u-space defined by

$$\frac{dx}{dt} = f(x, y, u), \; \frac{dy}{dt} = g(x, y, u), \quad \frac{du}{dt} = h(x, y, u). \tag{1.12}$$

We will show that a solution $u(x, y)$ of a quasi-linear equation may be interpreted as a surface made up of characteristics. This observation can be used to obtain solutions containing given (noncharacteristic) curves, and hence provides a way of solving the Cauchy problem when the partial differential equation is quasi-linear.

We need two observations.

Observation One Suppose $u = \varphi(x, y)$ is a solution of equation 1.11 defining a surface Σ, and that $P_0 : (x_0, y_0, u_0)$ is a point on Σ, so $u_0 = \varphi(x_0, y_0)$. Then the characteristic passing through P_0 lies entirely on Σ.

To see why this is true, suppose the characteristic has parametric equations

$$x = x(t), \; y = y(t), \; u = u(t).$$

Then for some t_0,

$$x(t_0) = x_0, \; y(t_0) = y_0, \; u(t_0) = u_0.$$

Because this curve is characteristic, we can use equations 1.12 to compute

$$\frac{dt}{dt}\varphi(x(t), y(t)) = \varphi_x \frac{dx}{dt} + \varphi_y \frac{dy}{dt}$$

$$= \varphi_x f(x, y, u) + \varphi_y g(x, y, u) = h(x, y, u) = \frac{du}{dt}.$$

Therefore

$$u(t) = \varphi(x(t), y(t)) + k$$

for some constant k. But P_0 is on the surface, so

$$\varphi(x(t_0), y(t_0)) = u_0 = u(t_0)$$

implies that $k = 0$. Therefore

$$u(t) = \varphi(x(t), y(t))$$

and the characteristic lies on Σ.

Observation Two If we begin with an arbitrary (but noncharacteristic) curve Γ and construct the family of characteristics passing through points of Γ, as in Figure 1.2, then the resulting surface Σ is the graph of a solution of the partial differential equation.

To see why this is true, assume that Σ is the graph of $u = \varphi(x, y)$. We want to show that φ is a solution of equation 1.11.

Suppose Γ is parameterized by

$$x = x(s),\ y = y(s),\ z = z(s).$$

At any (x, y, u) on Σ,

$$\frac{dx}{ds} = f(x, y, u),\ \frac{dy}{ds} = g(x, y, u),\ \frac{du}{ds} = h(x, y, u)$$

because the surface is made up of characteristics. Then

$$\frac{du}{ds} = h(x, y, u) = \varphi_x \frac{dx}{ds} + \varphi_y \frac{dy}{ds} = f\varphi_x + g\varphi_y$$

so φ is a solution.

FIGURE 1.2 A solution surface formed by constructing characteristics through points of Γ.

These observations suggest the *method of characteristics* for solving the Cauchy problem when the partial differential equation is quasi-linear. Suppose we want the solution of equation 1.11 assuming prescribed values on a given curve Γ that is not characteristic. Construct the characteristic through each point of Γ. This defines a surface in three space, and this surface is the graph of the solution of the Cauchy problem.

This strategy also suggests why we do not want to specify data along a characteristic C. If we did so, then the characteristic through each point of C would be just C itself, not a surface.

Here are two illustrations of the method of characteristics.

Example 5 We want the solution of

$$yu_x - xu_y = e^u$$

that passes through the curve Γ given by $y = \sin(x)$, $u = 0$. This means that we require

$$u(x, \sin(x)) = 0.$$

The characteristics of this partial differential equation are specified by

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x, \frac{du}{dt} = e^u.$$

From the first two of these equations we can write

$$\frac{dy}{dx} = -\frac{x}{y},$$

or

$$ydy + xdx = 0,$$

with general solution (in terms of t)

$$x = a\cos(t) + b\sin(t),\ y = b\cos(t) - a\sin(t),$$

with a and b constant. From $du/dt = e^u$ we obtain

$$-e^{-u} = t + c.$$

The characteristics therefore have parametric representation

$$x = a\cos(t) + b\sin(t),\ y = b\cos(t) - a\sin(t),\ e^{-u} = c - t.$$

Parameterize Γ as

$$x = s, \quad y = \sin(s),\ u = 0.$$

We use s as parameter on Γ to distinguish between points on Γ and points on characteristics.

We want to construct a characteristic through each point of Γ (as in Figure 1.2). Let $P:(s, \sin(s), 0)$ be a point of Γ and suppose a characteristic passes through P when $t = 0$ (this is just a scaling of the parameter). Then at $t = 0$,

$$x = a = s,\ y = b = \sin(s),\ \text{and } e^0 = 1 = 0 + c,$$

giving us $a = s$, $b = \sin(s)$, and $c = 1$ at this point of intersection of Γ with P. Therefore, the characteristic intersecting Γ at P has parametric equations

$$x = s\cos(t) + \sin(s)\sin(t),\ y = \sin(s)\cos(t) - s\sin(t),\ e^{-u} = 1 - t.$$

Now eliminate t and s from these equations. From the first two equations,

$$s = x\cos(t) - y\sin(t)$$

and

$$\sin(s) = y\cos(t) + x\sin(t).$$

Therefore

$$\sin(x\cos(t) - y\sin(t)) = y\cos(t) + x\sin(t). \tag{1.13}$$

But $e^{-u} = 1 - t$ implies that $t = 1 - e^{-u}$. Substitute this into equation (1.13) to get

$$\sin(x\cos(1 - e^{-u}) - y\sin(1 - e^{-u})) = y\cos(1 - e^{-u}) + x\sin(1 - e^{-u})).$$

This equation implicitly defines the solution of the Cauchy problem. It is easy to check that $y = \sin(x)$, $u = 0$ satisfies this equation.

Figure 1.3(a) shows a graph of part of the surface (solution), and Figure 1.3(b) shows the same surface from a different perspective. ■

Example 6 Consider the quasi-linear equation

$$xu_x + yu_y = \sec(u).$$

We would like the solution passing through

$$\Gamma: x = s^2,\ y = \sin(s),\ u = 0.$$

The characteristics satisfy

$$\frac{dx}{dt} = x,\ \frac{dy}{dt} = y,\ \frac{du}{dt} = \sec(u),$$

which have solutions defined by

$$x = Ae^t,\ y = Be^t,\ \sin(u) = t + c.$$

As in Example 5, we want to construct the characteristic through each point

FIGURE 1.3(a) Solution of $yu_x - xu_y = e^u$ containing the curve $y = \sin(x)$, $u = 0$ (Example 5).

FIGURE 1.3(b) Another perspective of the solution surface shown in Figure 1.3(a) (Example 5).

of Γ. Suppose a characteristic passes through Γ at $P:(s^2, \sin(s), 0)$ at $t = 0$. Then

$$x = A = s^2, \; y = B = \sin(s), \; \sin(0) = 0 = 0 + c,$$

so

$$A = s^2, \; B = \sin(s) \text{ and } c = 0$$

at this point. Then

$$x = s^2 e^t, \; y = \sin(s)e^t \text{ and } \sin(u) = t.$$

We want to eliminate s and t from these equations. Since $t = \sin(u)$, then

$$y = \sin(s)e^{\sin(u)}$$

so

$$\sin(s) = ye^{-\sin(u)}$$

and

$$s = \arcsin(ye^{-\sin(u)}).$$

Then

$$x = e^{\sin(u)}[\arcsin(ye^{-\sin(u)})]^2.$$

This equation implicitly defines $u(x, y)$ such that $u(s^2, \sin(s)) = 0$—that is, the solution surface contains the data curve Γ. Figure 1.4(a) shows a graph of Γ, and Figure 1.4(b) part of the solution surface. ■

EXERCISE 12 For each of the following, use the method of characteristics to find a solution of the partial differential equation that passes through the given curve Γ. The solution may be implicitly defined. Graph part of the solution surface.

1. $xu_x + yu_y = \sec(u)$; Γ is the curve defined by $y = x^3$, $u = 0$.
2. $u_x - xu_y = 4$; Γ is given by $y = 4x$, $u = 0$.
3. $u_x - y^2u_y = 1$; Γ is given by $y = x^2 + 2$; $u = 0$.
4. $u_x - y^3u_y = \sec(u)$; Γ is given by $y = x^2$, $u = 0$.
5. $u_x + yu_y = u$; Γ is given by $y = 1 - x$, $u = 1$.
6. $u_x + y^2u_y = \cos(u)$; Γ is given by $x = y^2$, $u = 0$.
7. $u_x - u_y = u^2$; Γ is given by $y = 2x - 1$, $u = 4$.
8. $x^3u_x - yu_y = u$; Γ is given by $x = y^2 - 1$, $u = 0$.
9. $u_x - y^2u_y = u$; Γ is given by $y = 1 - x^2$, $u = 2$.
10. $xu_x + u_y = e^u$; Γ is given by $y = x - 1$, $u = 0$.

FIGURE 1.4(a) Graph of the curve $x = t^2$, $y = \sin(t)$, $u = 0$ (Example 6).

FIGURE 1.4(b) Solution of $xu_x + yu_y = \sec(u)$ containing the curve shown in Figure 1.4(a).

EXERCISE 13 Use the method of characteristics to show that the solution of the problem

$$uu_x + u_y = 0$$

$$u(x, 0) = f(x)$$

is defined implicitly by the equation

$$u(x, y) = f(x - u(x, y)y).$$

Hint: Think of Γ as the curve $x = s$, $y = s$, $u = f(s)$.

Next use the idea of Exercise 4 to solve this problem. Compare these solutions.