Section 6.9

In each of the problems below, obtain the solution of the Dirichlet problem for the upper half − plane, using the given $f(x)$.

1] $f(x)=\begin{cases}0, & x<0 \text{ and } x>1\\ x & 0\le x\le 1\end{cases}$

4] $f(x)=e^{-|x|}$

Solution

Problem 1

$$f(x)=\begin{cases}0, & x<0 \text{ and } x>1\\ x & 0\le x\le 1\end{cases}$$

$$\text{Using } u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{y^2+(\xi-x)^2}\,d\xi$$

$$u(x,y)=\frac{y}{\pi}[\int_{-\infty}^{0}0+\int_{0}^{1}\frac{\xi}{y^2+(\xi-x)^2}d\xi\;+\;\int_{1}^{\infty}0$$

$$\therefore u(x,y)=\frac{y}{\pi}\int_{0}^{1}\frac{\xi}{y^2+(\xi-x)^2}d\xi$$

$$let\ \eta=\xi-x\ \Rightarrow d\eta=d\xi,$$

$$for\ \xi=1\,,\eta=1-x\ and\ for\ \xi=0,\eta=-x$$

$$\therefore\ u(\xi-\eta,y)=\frac{y}{\pi}\int_{-x}^{1-x}\frac{\eta+x}{y^2+\eta^2}d\eta$$

$$=\frac{y}{\pi}\left[\int_{-x}^{1-x}\frac{\eta}{y^2+\eta^2}d\eta+\int_{-x}^{1-x}\frac{x}{y^2+\eta^2}d\eta\right]\ldots..(1)$$

$$For\ the\ first\ part\ i.e\ \int_{-x}^{1-x}\frac{\eta}{y^2+\eta^2}d\eta$$

$$let\ t=\eta^2$$

$$\therefore dt=\ 2\eta d\eta$$

$$\therefore\int\frac{\eta}{y^2+\eta^2}d\eta=\int\frac{\eta}{y^2+t}\frac{dt}{2\eta}$$

$$= \frac{1}{2}\int \frac{dt}{y^2+t} = \frac{1}{2}\ln(y^2+t)$$

$$\therefore \int_{-x}^{1-x} \frac{\eta}{y^2+\eta^2}\, d\eta = \frac{1}{2}\ln(y^2+\eta^2)\,\Big|_{-x}^{1-x}$$

$$= \frac{1}{2}\left[\ln(y^2+(1-x)^2) - \ln(y^2+x^2)\right]$$

$$= \frac{1}{2}\ln\left[\frac{y^2+(1-x)^2}{(y^2+x^2)}\right] \ldots\ldots(2)$$

Similarly, for the 2nd part of (1)

$$i.e \int_{-x}^{1-x} \frac{x}{y^2+\eta^2}\, d\eta$$

$$= x\int_{-x}^{1-x} \frac{1}{y^2\left(1+\left(\frac{\eta}{y}\right)^2\right)}\, d\eta$$

$$\frac{x}{y^2}\int_{-x}^{1-x} \frac{d\eta}{\left(1+\left(\frac{\eta}{y}\right)^2\right)}$$

$$= \frac{x}{y^2}\, y\tan^{-1}\left(\frac{\eta}{y}\right)\Big|_{-x}^{1-x}$$

$$= \frac{x}{y}\left[\tan^{-1}\left(\frac{1-x}{y}\right) - \tan^{-1}\left(\frac{-x}{y}\right)\right]$$

$$= \frac{x}{y}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right] \ldots\ldots\ldots(3)$$

Thus, from (1), (2) *and* (3), *we have*

$$\therefore u(x,y) = \frac{y}{\pi}\left[\frac{1}{2}\ln\left[\frac{y^2+(1-x)^2}{(y^2+x^2)}\right] + \frac{x}{y}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right]\right]$$

$$= \frac{y}{2\pi}\ln\left[\frac{y^2+(1-x)^2}{(y^2+x^2)}\right] + \frac{x}{y}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)\right]$$

Problem 4

$f(x) = e^{-|x|}$

$Using\ u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{y^2+(\xi-x)^2}\ d\xi$

$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{e^{-|\xi|}}{y^2+(\xi-x)^2}$

$= \frac{y}{\pi}\int_{-\infty}^{0}\frac{e^{\xi}}{y^2+(\xi-x)^2}\,d\xi + \int_{0}^{\infty}\frac{e^{-\xi}}{y^2+(\xi-x)^2}\,d\xi$

This integral is very complicated to solve. Hence, we use

$u(x,y) = \int_0^{\infty}[a_\omega\cos(\omega\xi) + b_\omega sin(\omega\xi)]\ d\omega \ldots\ldots\ldots(4)$

$where\ \ a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi$

$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi$

$\Rightarrow a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|\xi|}\cos(\omega\xi)\,d\xi$

$= \frac{1}{\pi}\left[\int_{-\infty}^{0} e^{\xi}\cos(\omega\xi)\,d\xi + \int_0^{\infty} e^{-\xi}\cos(\omega\xi)\,d\xi\right]\ldots\ldots.(5)$

Using integration by parts for each, we have

$\int_{-\infty}^{0} e^{\xi}\cos(\omega\xi)\,d\xi$

$u = e^{\xi} \quad dv = \cos(\omega\xi)\,d\xi$

$du = e^{\xi}\,d\xi \quad v = \frac{\sin(\omega\xi)}{\omega}$

$\therefore\ \int_{-\infty}^{0} e^{\xi}\cos(\omega\xi)\,d\xi = \frac{e^{\xi}\sin(\omega\xi)}{\omega}\Big|_{-\infty}^{0} - \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi}\sin(\omega\xi)\,d\xi$

$for\ \int_{-\infty}^{0} e^{\xi}\sin(\omega\xi)\,d\xi$

$let\ u = e^{\xi} \quad dv = \sin(\omega\xi)\,d\xi$

$du = e^{\xi} d\xi \quad v = \frac{-\cos(\omega\xi)}{\omega}$

$\therefore \int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = \frac{-e^{\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0} + \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi$

$\therefore \int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi = \frac{e^{\xi}\sin(\omega\xi)}{\omega} \Big|_{-\infty}^{0} - \frac{1}{\omega}\Big[\frac{-e^{\xi}\cos(\omega\xi)}{\omega}\Big]\Big|_{-\infty}^{0}$

$+ \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi$

$= \frac{e^{\xi}\sin(\omega\xi)}{\omega} \Big|_{-\infty}^{0} + \left[\frac{e^{\xi}\cos(\omega\xi)}{\omega^2}\right]\Big|_{-\infty}^{0} - \frac{1}{\omega^2}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi$

$\therefore \left(1 + \frac{1}{\omega^2}\right)\int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi = 0 - \lim_{\xi\to-\infty} \frac{e^{\xi}\sin(\omega\xi)}{\omega} + \frac{1}{\omega^2}$

$- \lim_{\xi\to-\infty} \frac{e^{\xi}\cos(\omega\xi)}{\omega^2}$

$\frac{1+\omega^2}{\omega^2}\int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi = \frac{1}{\omega^2} - \lim_{\xi\to-\infty} \frac{e^{\xi}\sin(\omega\xi)}{\omega^2} - \lim_{\xi\to-\infty} \frac{e^{\xi}\cos(\omega\xi)}{\omega^2}.(6)$

$Since\ \lim_{\xi\to-\infty} \sin(\omega\xi)\ and\ \lim_{\xi\to-\infty} \cos(\omega\xi)\ are\ not\ define, we\ can\ use$

$Squeeze\ Theorem\ to\ find\ both\ limit\ as\ \xi \to -\infty$

$Using\ Squeeze\ Theorem, we\ have$

$-1 \le \sin(\omega\xi) \le 1$

$\Rightarrow \frac{-e^{\xi}}{\omega} \le \frac{e^{\xi}\sin(\omega\xi)}{\omega} \le \frac{e^{\xi}}{\omega}$

$-\lim_{\xi\to-\infty} \frac{e^{\xi}}{\omega} \le \lim_{\xi\to-\infty} \frac{e^{\xi}\sin(\omega\xi)}{\omega} \le \lim_{\xi\to-\infty} \frac{e^{\xi}}{\omega}$

$0 \le \lim_{\xi\to-\infty} \frac{e^{\xi}\sin(\omega\xi)}{\omega} \le 0$

$\Rightarrow \lim_{\xi\to-\infty} \frac{e^{\xi}\sin(\omega\xi)}{\omega} = 0$

Similarly,

$$-1 \leq \cos(\omega\xi) \leq 1$$

$$\Rightarrow \frac{-e^{\xi}}{\omega} \leq \frac{e^{\xi}\cos(\omega\xi)}{\omega} \leq \frac{e^{\xi}}{\omega}$$

$$-\lim_{\xi\to-\infty}\frac{e^{\xi}}{\omega} \leq \lim_{\xi\to-\infty}\frac{e^{\xi}\cos(\omega\xi)}{\omega} \leq \lim_{\xi\to-\infty}\frac{e^{\xi}}{\omega}$$

$$0 \leq \lim_{\xi\to-\infty}\frac{e^{\xi}\cos(\omega\xi)}{\omega} \leq 0$$

$$\Rightarrow \lim_{\xi\to-\infty}\frac{e^{\xi}\cos(\omega\xi)}{\omega} = 0$$

$\therefore$ *from* (6), *we have*

$$\frac{1+\omega^2}{\omega^2}\int_{-\infty}^{0} e^{\xi}\cos(\omega\xi)\,d\xi = \frac{1}{\omega^2}$$

$$\therefore \int_{-\infty}^{0} e^{\xi}\cos(\omega\xi)\,d\xi = \frac{1}{\omega^2}.\frac{\omega^2}{1+\omega^2} = \frac{1}{1+\omega^2} \dots\dots\dots (7)$$

Similarly, using integration by parts for

$$\int_0^{\infty} e^{-\xi}\cos(\omega\xi)\,d\xi$$

$$u = e^{-\xi} \qquad dv = \cos(\omega\xi)\,d\xi$$

$$du = -e^{-\xi}d\xi \qquad v = \frac{\sin(\omega\xi)}{\omega}$$

$$\therefore \int_0^{\infty} e^{-\xi}\cos(\omega\xi)\,d\xi = \frac{e^{-\xi}\sin(\omega\xi)}{\omega}\Big|_0^{\infty} + \frac{1}{\omega}\int_0^{\infty} e^{-\xi}\sin(\omega\xi)\,d\xi$$

for $\int_0^{\infty} e^{-\xi}\sin(\omega\xi)\,d\xi$

let $u = e^{-\xi} \qquad dv = \sin(\omega\xi)\,d\xi$

$$du = -e^{-\xi}d\xi \qquad v = \frac{-\cos(\omega\xi)}{\omega}$$

$$\therefore \int_0^{\infty} e^{-\xi}\sin(\omega\xi)\,d\xi = \frac{-e^{-\xi}\cos(\omega\xi)}{\omega}\Big|_0^{\infty} - \frac{1}{\omega}\int_0^{\infty} e^{-\xi}\cos(\omega\xi)\,d\omega$$

$$\therefore \int_0^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi = \frac{e^{-\xi}\sin(\omega\xi)}{\omega} \Big|_0^{\infty} + \frac{1}{\omega}\Big[\frac{-e^{-\xi}\cos(\omega\xi)}{\omega}\Big|_0^{\infty}$$

$$-\frac{1}{\omega}\int_0^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi\Big]$$

$$= \frac{e^{-\xi}\sin(\omega\xi)}{\omega}\Big|_{-\infty}^{0} - \frac{e^{-\xi}\cos(\omega\xi)}{\omega^2}\Big|_{-\infty}^{0} - \frac{1}{\omega^2}\int_0^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi$$

$$\left(1+\frac{1}{\omega^2}\right)\int_0^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi = \left[\lim_{\xi\to\infty} \frac{e^{-\xi}\sin(\omega\xi)}{\omega} - \frac{e^{0}\sin(0)}{\omega}\right]$$

$$-\left[\lim_{\xi\to\infty} \frac{e^{-\xi}\cos(\omega\xi)}{\omega^2} - \frac{e^{0}\cos(0)}{\omega^2}\right]$$

$$\frac{1+\omega^2}{\omega^2}\int_0^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi = \frac{1}{\omega}\lim_{\xi\to\infty} e^{-\xi}\sin(\omega\xi) - 0$$

$$-\frac{1}{\omega^2}\lim_{\xi\to\infty} e^{-\xi}\cos(\omega\xi) + \frac{1}{\omega^2} \ldots\ldots (8)$$

Since $\lim_{\xi\to\infty}\cos(\omega\xi)$ *and* $\lim_{\xi\to\infty}\sin(\omega\xi)$ *are not define, we can use*

Squeeze Theorem to evaluate the limit

$\therefore$ *Using Squeeze Theorem,*

$$-1 \le \sin(\omega\xi) \le 1$$

$$\Rightarrow \frac{-e^{-\xi}}{\omega} \le \frac{e^{-\xi}\sin(\omega\xi)}{\omega} \le \frac{e^{-\xi}}{\omega}$$

$$-\frac{1}{\omega}\lim_{n\to\infty} e^{-\xi} \le \lim_{\xi\to\infty} \frac{e^{-\xi}\sin(\omega\xi)}{\omega} \le \frac{1}{\omega}\lim_{\xi\to\infty} e^{-\xi}$$

$$0 \le \lim_{\xi\to\infty} \frac{e^{-\xi}\sin(\omega\xi)}{\omega} \le 0$$

$$\Rightarrow \lim_{\xi\to\infty} \frac{e^{-\xi}\sin(\omega\xi)}{\omega} = 0$$

$Similarly,$

$$-1 \leq \cos(\omega\xi) \leq 1$$

$$\Rightarrow \frac{-e^{-\xi}}{\omega} \leq \frac{e^{-\xi}\cos(\omega\xi)}{\omega} \leq \frac{e^{-\xi}}{\omega}$$

$$-\frac{1}{\omega}\lim_{\xi\to\infty} e^{-\xi} \leq \lim_{\xi\to\infty} \frac{e^{-\xi}\cos(\omega\xi)}{\omega} \leq \frac{1}{\omega}\lim_{\xi\to\infty} e^{-\xi}$$

$$0 \leq \lim_{\xi\to\infty} \frac{e^{-\xi}\cos(\omega\xi)}{\omega} \leq 0$$

$$\Rightarrow \lim_{\xi\to\infty} \frac{e^{-\xi}\cos(\omega\xi)}{\omega} = 0$$

$\therefore$ *from* (8) *we have,*

$$\frac{1+\omega^2}{\omega^2}\int_0^\infty e^{-\xi}\cos(\omega\xi)\,d\xi = \frac{1}{\omega^2}$$

$$\therefore \int_0^\infty e^{-\xi}\cos(\omega\xi)\,d\xi = \frac{1}{\omega^2}.\frac{\omega^2}{1+\omega^2} = \frac{1}{1+\omega^2} \ldots\ldots\ldots (9)$$

substitute(7) *and* (9) *into* (5), *we have*

$$a_\omega = \frac{1}{\pi}\left[\frac{1}{1+\omega^2} + \frac{1}{1+\omega^2}\right] = \frac{1}{\pi}\left[\frac{2}{1+\omega^2}\right] = \frac{2}{\pi(1+\omega^2)}$$

Similarly, we solve for b_ω

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} e^{\xi}\sin(\omega\xi)\,d\xi + \int_0^\infty e^{-\xi}\sin(\omega\xi)\,d\xi\right] \ldots\ldots (10)$$

Using integration by parts for

$$\int_{-\infty}^{0} e^{\xi}\sin(\omega\xi)\,d\xi$$

$$u = e^{\xi} \qquad dv = \sin(\omega\xi)\,d\xi$$

$$du = e^{\xi}\,d\xi \qquad v = \frac{-\cos(\omega\xi)}{\omega}$$

$$\therefore \int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = \frac{-e^{\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0} + \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi$$

$for \int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi$

$let\ u = e^{\xi} \quad dv = \cos(\omega\xi)\, d\omega$

$du = e^{\xi} d\xi \quad v = \frac{\sin(\omega\xi)}{\omega}$

$$\therefore \int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi = \frac{e^{\xi}\sin(\omega\xi)}{\omega} \Big|_{-\infty}^{0} - \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi$$

$$\therefore \int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = \frac{-e^{\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0} + \frac{1}{\omega}\Big[\frac{e^{\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0}$$

$$- \frac{1}{\omega}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi\Big]$$

$$= \frac{-e^{\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0} + \frac{e^{\xi}\sin(\omega\xi)}{\omega^2} \Big|_{-\infty}^{0} - \frac{1}{\omega^2}\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi$$

$$\left(1 + \frac{1}{\omega^2}\right)\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = -\frac{1}{\omega} e^{\xi}\cos(\omega\xi) \Big|_{-\infty}^{0} + \frac{1}{\omega^2} e^{\xi}\sin(\omega\xi) \Big|_{-\infty}^{0}$$

$$\frac{1+\omega^2}{\omega^2}\int_{-\infty}^{0} e^{\xi} \cos(\omega\xi)\, d\xi = \frac{-1}{\omega}\big[1 - \lim_{\xi\to-\infty} e^{\xi}\cos(\omega\xi)\big]$$

$$+ \frac{1}{\omega^2}\left[0 - \lim_{\xi\to-\infty} e^{\xi}\sin(\omega\xi)\right]$$

$$= \frac{-1}{\omega} + \frac{1}{\omega}\lim_{\xi\to-\infty} e^{\xi}\cos(\omega\xi) - \frac{1}{\omega^2}\lim_{\xi\to-\infty} e^{\xi}\sin(\omega\xi)$$

Using Squeeze Theorem for both $\lim_{\xi\to-\infty} e^{\xi}\cos(\omega\xi)$ *and*

$\lim_{\xi\to-\infty} e^{\xi} \sin(\omega\xi)$, *we get the same result as in the first part i.e*

$$\lim_{\xi\to-\infty} e^{\xi}\cos(\omega\xi) = 0 \ and \lim_{\xi\to-\infty} e^{\xi} \sin(\omega\xi) = 0$$

$$\therefore \left(\frac{1+\omega^2}{\omega^2}\right)\int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = \frac{-1}{\omega}$$

$$\therefore \int_{-\infty}^{0} e^{\xi} \sin(\omega\xi)\, d\xi = \frac{-\omega}{1+\omega^2} \ldots \ldots (11)$$

Similarly, Using integration by parts for

$$\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi$$

$$u = e^{-\xi} \qquad dv = \sin(\omega\xi)\, d\xi$$

$$du = -e^{-\xi} d\xi \qquad v = \frac{-\cos(\omega\xi)}{\omega}$$

$$\therefore \int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi = \frac{-e^{-\xi}\cos(\omega\xi)}{\omega} \Big|_{-\infty}^{0} - \frac{1}{\omega}\int_{0}^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi$$

$$for \int_{0}^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi$$

$$let\ u = e^{-\xi} \qquad dv = \cos(\omega\xi)\, d\xi$$

$$du = -e^{-\xi} d\xi \qquad v = \frac{\sin(\omega\xi)}{\omega}$$

$$\therefore \int_{0}^{\infty} e^{-\xi} \cos(\omega\xi)\, d\xi = \frac{e^{-\xi}\sin(\omega\xi)}{\omega} \Big|_{0}^{\infty} + \frac{1}{\omega}\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi$$

$$\therefore \int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi = \frac{-e^{-\xi}\cos(\omega\xi)}{\omega} \Big|_{0}^{\infty} - \frac{1}{\omega}\Big[\frac{e^{-\xi}\sin(\omega\xi)}{\omega} \Big|_{0}^{\infty} + \frac{1}{\omega}\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi\Big]$$

$$= \frac{-e^{-\xi}\cos(\omega\xi)}{\omega} \Big|_{0}^{\infty} - \left[\frac{1}{\omega^2} e^{-\xi} \sin(\omega\xi) \Big|_{0}^{\infty} - \frac{1}{\omega^2}\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi \right]$$

$$\left(1 + \frac{1}{\omega^2}\right)\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi = -\left[\lim_{\xi\to\infty} \frac{e^{-\xi}\cos(\omega\xi)}{\omega} - \frac{1}{\omega}\right] - \frac{1}{\omega^2}\left[\lim_{\xi\to\infty} e^{-\xi} \sin(\omega\xi) - 0\right]$$

$$\frac{1+\omega^2}{\omega^2}\int_{0}^{\infty} e^{-\xi} \sin(\omega\xi)\, d\xi = -\frac{1}{\omega}\lim_{\xi\to\infty} e^{-\xi} \cos(\omega\xi) + \frac{1}{\omega} - \frac{1}{\omega^2}\lim_{\xi\to\infty} e^{-\xi}\sin(\omega\xi)$$

$Using\ Squeeze\ Theorem\ for\ both\ \lim_{\xi\to\infty} e^{-\xi}\cos(\omega\xi)\ \ and\ \lim_{\xi\to\infty} e^{-\xi}\sin(\omega\xi),$

$we\ get\ thesame\ as\ in\ the\ first\ part\ i.e$

$$\lim_{\xi\to\infty} e^{-\xi}\cos(\omega\xi)\ = 0\ \ and\ \lim_{\xi\to\infty} e^{-\xi}\sin(\omega\xi) = 0$$

$$\therefore \frac{1+\omega^2}{\omega^2}\int_0^\infty e^{-\xi}\sin(\omega\xi)\,d\xi = \frac{1}{\omega}$$

$$\therefore \int_0^\infty e^{-\xi}\sin(\omega\xi)\,d\xi = \frac{\omega}{1+\omega^2} \ldots\ldots.(12)$$

$putting\ (11)\ and\ (12)\ into\ (10), we\ have$

$$b_\omega = \frac{1}{\pi}\left[\frac{-w}{1+\omega^2} + \frac{w}{1+\omega^2}\right] = 0$$

$Putting\ the\ value\ of\ a_\omega\ and\ b_\omega\ into\ (4), we\ have$

$$u(x,y) = \int_0^\infty \left[\frac{2}{\pi(1+\omega^2)}\cos(\omega\xi) + 0\right]d\omega$$

$$\Rightarrow u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\omega\xi)}{1+\omega^2}\,d\omega$$

<u>*SECTION 6.10*</u>

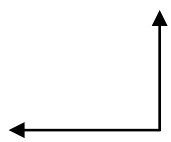
1] *Find a bounded solution of the Dirichlet problem for the left quarter − plane:*

$\nabla^2 u = 0 \; for \; x < 0, y > 0$

$u(0, y) = 0 \; for \; y > 0$

$u(x, 0) = f(x) \quad for \; x < 0$

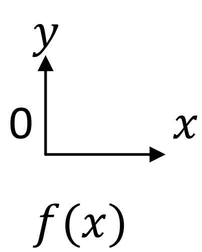
<u>*Solution*</u>

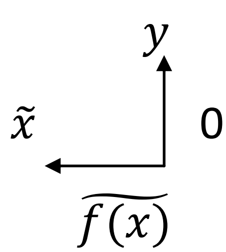


Here, we need to transform from right quarter − plane to left − quarter plane.

$\widetilde{f(x)} = -f(x)$

i.e $\tilde{x} = -x$





$u_{lp}(\tilde{x}, y) = u_{hp}(-x, y)$

$$u_{hp}(x, y) = \frac{y}{\pi} \int_0^{\infty} \left(\frac{1}{y^2 + (\xi - x)^2} - \frac{1}{y^2 + (\xi + x)^2} \right) f(\xi) d\xi$$

⇓

$$u_{Lp}(\tilde{x}, y) = \frac{y}{\pi} \int_{-\infty}^{0} \left(\frac{1}{y^2 + (\xi + \tilde{x})^2} - \frac{1}{y^2 + (\xi - \tilde{x})^2} \right) - \widetilde{f(\xi)} d\xi$$

$$= \frac{y}{\pi} \int_{-\infty}^{0} \left(\frac{1}{y^2 + (\xi - \tilde{x})^2} - \frac{1}{y^2 + (\xi + \tilde{x})^2} \right) \widetilde{f(\xi)} d\xi$$

$\therefore$ *The bounded solution for the left quarter − plane is*

$$u_{Lp}(x,y) = \frac{y}{\pi}\int_{-\infty}^{0}\left(\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right) f(\xi)d\xi$$