$Section\ 6.2$

$Produce\ harmonic\ functions\ u\ and\ v\ for\ the\ following$

$by\ writing\ the\ given\ complex\ function\ in\ the\ form$

$f(z) = u(x, y) + iv(x, y).$

$3)\ f(z) = zcos(z)$

$4)\ f(z) = sin^2(z)$

$\underline{SOLUTION}$

$\underline{Problem\ 3}$

$f(z) = zcos(z)$

$where\ z = x + iy$

$\underline{Solution}$

$f(z) = (x + iy)cos(x + iy)$

$= (x + iy)[cos(x)cos(iy) - sin(x)sin(iy)]$

$= (x + iy)[cos(x)cosh(y) - isin(x)sinh(y)]$

$since\ cos(iy) = cosh(y)\ and\ sin(iy) = isinh(y).$

$= xcos(x)cosh(y) + ysin(x)sinh(y)$

$+iycos(x)cosh(y) - ix\ sin(x)sinh(y)$

$= xcos(x)cosh(y) + ysin(x)sinh(y)$

$+i[ycos(x)cosh(y) - x\ sin(x)sinh(y)$

$which\ is\ the\ harmonic\ function\ in\ the\ form$

$f(z) = u(x, y) + iv(x, y)$

$with\ u(x, y) = xcos(x)cosh(y) + ysin(x)sinh(y)\ and$

$v(x, y) = ycos(x)cosh(y) - xsin(x)sinh(y).$

Verification

$$U_x = \big(\cos(x) - x\sin(x)\big)\cosh(y) + y\cos(x)\sinh(y)$$

$$U_{xx} = \big(-\sin(x) - \sin x - x\cos(x)\big)\cosh(y) - y\sin(x)\sinh(y)$$

$$= (-2\sin(x) - x\cos(x))\cosh(y) - \ y\sin(x)\sinh(y)$$

$$U_y = x\cos(x)\sinh(y) + \big(y\cosh(y) + \sinh(y)\big)\sin(x)$$

$$U_{yy} = x\cos(x)\cosh(y) + (\cosh(y) + y\sinh(y) + \cosh(y))\sin(x)$$

$$= x\cos(x)\cosh(y) + \big(2\cosh(y) + y\sinh(y)\big)\sin(x)$$

$$U_{xx} + U_{yy} = -2\sin(x)\cosh(y) - x\cos(x)\cosh(y) - y\sin(x)\sinh(y)$$

$$+x\cos(x)\cosh(y) + 2\sin(x)\cosh(y) + y\sin(x)\sinh(y)$$

$$\therefore U_{xx} + U_{yy} = 0 \Rightarrow Harmonic.$$

Similarly,

$$v_x = -y\sin(x)\cosh(y) - \big(\sin(x) + x\cos(x)\big)\sinh(y)$$

$$v_{xx} = -y\cos(x)\cosh(y) - \big(\cos(x) + \cos(x) - x\sin(x)\big)\sinh(y)$$

$$v_{xx} = -y\cos(x)\cosh(y) - \big(2\cos(x) - x\sin(x)\big)\sinh(y)$$

$$v_y = \big(\cosh(y) + y\sinh(y)\big)\cos(x) - x\sin(x)\cosh(y)$$

$$v_{yy} = (\sinh(y) + \sinh(y) + y\cosh(y))\cos(x) - x\sin(x)\sinh(y)$$

$$v_{yy} = (2\sinh(y) + y\cosh(y))\cos(x) - x\sin(x)\sinh(y)$$

$$v_{xx} + v_{yy} = -y\cos(x)\cosh(y) - 2\cos(x)\sinh(y) + x\sin(x)\sinh(y)$$

$$+2\sinh(y)\cos(x) + y\cosh(y)\cos(x) - x\sin(x)\sinh(y)$$

$$\therefore \ v_{xx} + v_{yy} = 0 \Rightarrow Harmonic$$

Hence, the function itself is harmonic.

Problem 4

$f(z) = sin^2(z)$

$where\ z = x + iy$

Solution

$$f(z) = sin^2(z) = \frac{1-cos2z}{2} = \frac{1}{2}\,(1 - cos(2z))$$

$$= \frac{1}{2}\,\big(1 - cos(2x + 2iy)\big)$$

$$= \frac{1}{2}\,[1 - \{cos(2x)cos(2iy) - sin(2x)sin(2iy)\}]$$

$$= \frac{1}{2}\,[1 - \{cos(2x)cosh(2y) - isin(2x)sinh(2y)\}]$$

$$\therefore f(z) = \frac{1}{2}\,[1 - \{cos(2x)cosh(2y) - isin(2x)sinh(2y)\}]$$

$Which\ is\ the\ harmonic\ function\ in\ the\ form$

$$f(z) = u(x,y) + i(x,y)$$

$$with\ u(x,y) = \frac{1}{2}\big(1 - cos(2x)cosh(2y)\big)$$

$$and\ v(x,y) = \frac{1}{2}sin(2x)sinh(2y)$$

Verification

$$u_x = \frac{1}{2}[2sin(2x)cosh(2y)] = sin(2x)cosh(2y)$$

$$u_{xx} = 2cos(2x)cosh(2y)$$

$$u_y = \frac{1}{2}[-2cos(2x)sinh(2y)] = -cos(2x)sinh(2y)$$

$$u_{yy} = -2cos(2x\ )cosh(2y)$$

$$u_{xx} + u_{yy} = 2cos(2x)cosh(2y) - 2cos(2x\ )cosh(2y)$$

$$\therefore\ u_{xx} + u_{yy} = 0 \Rightarrow Harmonic$$

$Similarly,$

$$v_x = \frac{1}{2}[2cos(2x)sinh(2y)] = cos(2x)sinh(2y)$$

$$v_{xx} = -2sin(2x\ )sinh(2y)$$

$$v_y = \frac{1}{2}[2sin(2x)cosh(2y)] = sin(2x)cosh(2y)$$

$$v_{yy} = 2sin(2x\ )sinh(2y)$$

$$v_{xx} + v_{yy} = -2sin(2x\ )sinh(2y) + 2sin(2x\ )sinh(2y) = 0$$

$$\therefore v_{xx} + v_{yy} = 0 \Rightarrow Harmonic$$

$Hence, the\ function\ itself\ is\ harmonic.$

Problem 5

Let u be harmonic in a domain Ω in R^2.

Let a and b be real numbers and let Ω^* consist of all translation $(x+a, y+b)$ of points (x, y) in Ω. Define

$$v(x,y) = u(x-a, y-b)$$

for (x, y) in Ω^*. Prove that v is harmonic in Ω^*.

This means that translations of harmonic functions are harmonic functions.

Solution

To show that v is harmonic

Transformation

$v(x,y) = u(x-a, y-b) \;\; \Rightarrow x = x-a \;\text{ and }\; y = y-a$

$v_x = u_x, \quad v_y = u_y$

$v_{xx} = u_{xx}, \quad v_{yy} = u_{yy}$

$\therefore v_{xx} + v_{yy} = u_{xx} + u_{yy}.$

Since u is harmonic $\Rightarrow u_{xx} + u_{yy} = 0$

$\therefore v_{xx} + v_{yy} = 0.$ *Hence v is harmonic in* Ω^*

Thus, the translations of harmonic functions are harmonic function

Problem 6

Let θ be any real number and let

$$x^* = \cos(\theta)x + \sin(\theta)y$$

$$y^* = -\sin(\theta)x + \cos(\theta)y$$

a rotation about the origin in the plane.

Suppose that u is harmonic in Ω, and let Ω^* be obtained by applying the rotation to all points of Ω.

Let $w(x^*, y^*) = u(x, y)$. Show that w is harmonic in Ω^*.

This means that rotations take harmonic functions to harmonic functions.

Solutions

let $w(x^*, y^*) = u(x, y)$

To show that w is harmonic in Ω^*

Transformation

$$w(x^*, y^*) = u(x, y)$$

$$x^* = \cos(\theta)x + \sin(\theta)y \qquad y^* = -\sin(\theta)x + \cos(\theta)y$$

$$\therefore x^*_x = \cos(\theta), \ \ x^*_{xx} = 0 \qquad y^*_x = -\sin(\theta), \ \ y^*_{xx} = 0$$

$$x^*_y = \sin(\theta), \ \ x^*_{yy} = 0 \qquad y^*_y = \cos(\theta), \ \ y^*_{yy} = 0$$

$$u_x = w_{x^*}x^*_x + w_{y^*}y^*_x$$

where $x^*_x = \frac{\partial x^*}{\partial x}$ ***and*** $y^*_x = \frac{\partial y^*}{\partial x}$

$$u_x = \cos(\theta)w_{x^*} - \sin(\theta)w_{y^*}$$

$$u_{xx} = \cos(\theta)\left[w_{x^*x^*}x^*_x + w_{x^*y^*}y^*_x\right] - \sin(\theta)\left[w_{y^*x^*}x^*_x + w_{y^*y^*}y^*_x\right]$$

$$= cos(\theta)\left[cos(\theta)w_{x^*x^*} - sin(\theta)w_{x^*y^*}\right]$$

$$-sin(\theta)\left[cos(\theta)w_{y^*x^*} - sin(\theta)w_{y^*y^*}\right]$$

$$\therefore u_{xx} = cos^2(\theta)w_{x^*x^*} - sin(\theta)cos(\theta)w_{x^*y^*}$$

$$- sin(\theta)cos(\theta)w_{y^*x^*} + sin^2(\theta)w_{y^*y^*}$$

$$= cos^2(\theta)w_{x^*x^*} - 2sin(\theta)cos(\theta)w_{x^*y^*} + sin^2(\theta)w_{y^*y^*} \ldots. i$$

Similarly,

$$u_y = w_{x^*}{x^*}_y + w_{y^*}{y^*}_y$$

Where $${x^*}_y = \frac{\partial x^*}{\partial y} \text{ and } {y^*}_y = \frac{\partial y^*}{\partial y}$$

$$u_y = sin(\theta)w_{x^*} + cos(\theta)w_{y^*}$$

$$u_{yy} = sin(\theta)\left[w_{x^*x^*}{x^*}_y + w_{x^*y^*}{y^*}_y\right] + cos(\theta)\left[w_{y^*x^*}{x^*}_y + w_{y^*y^*}{y^*}_y\right]$$

$$= sin(\theta)\left[sin(\theta)w_{x^*x^*} + cos(\theta)w_{x^*y^*}\right]$$

$$+ cos(\theta)\left[sin(\theta)w_{y^*x^*} + cos(\theta)w_{y^*y^*}\right]$$

$$sin^2(\theta)w_{x^*x^*} + sin(\theta)cos(\theta)w_{x^*y^*} + sin(\theta)cos(\theta)w_{y^*x^*}$$

$$+ cos^2(\theta)w_{y^*y^*}$$

$$= sin^2(\theta)w_{x^*x^*} + 2sin(\theta)cos(\theta)w_{x^*y^*} + cos^2(\theta)w_{y^*y^*} \ldots\ldots ii$$

From i and ii, we have

$$u_{xx} + u_{yy} = (sin^2\theta + cos^2\theta)w_{x^*x^*} + 0 + (sin^2\theta + cos^2\theta)w_{y^*y^*}$$

Since $sin^2(\theta) + cos^2(\theta) = 1$

$$\therefore u_{xx} + u_{yy} = w_{x^*x^*} + w_{y^*y^*}$$

Since u is harmonic $\Rightarrow u_{xx} + u_{yy} = 0$

$\therefore\ w_{x^*x^*} + w_{y^*y^*} = 0$

Hence, w is harmonic in Ω^*

Thus, the relations take harmonic functions to harmonic functions.