Section 6.1

Problem 8

Show that $u_n(x,y) = e^{-\sqrt{n}} \sin(nx) e^{ny}$ ***is a solution of***

$\nabla^2 u(x,y) = 0 \;\; \forall x$ ***and*** $t > 0$

$u(x,0) = 0, U_y(x,0) = ne^{-\sqrt{n}} \sin(nx)$ ***for each positive integer*** n.

Prove that for any positive $y_0, u_n(x, y_0) \to \infty$ ***as*** $n \to \infty$

Solution

Since the question is wrong because

$u_n(x,y) = e^{-\sqrt{n}} \sin(nx) e^{ny}$ ***does not satisfy*** $u(x,0) = 0$.

Hence, I need to solve the laplace equation subjected to the given conditions to get the right solution to the problem.

Consider;

$$\nabla^2 u(x,y) = u_{xx} + u_{yy} = 0$$

$$\Rightarrow u_{xx} = -u_{yy}$$

$$\Rightarrow u_{xx} = i^2 u_{yy} \qquad *$$

$*$ ***is now wave equation. Hence, I can apply*** d' ***Alembert formula.***

$$u_n(x,y) = \frac{1}{2}[\varphi(x - cy) + \varphi(x + cy)] + \frac{1}{2c}\int_{x-cy}^{x+cy} \psi(s) ds$$

where $\varphi(x) = 0 \qquad \Rightarrow \varphi(x - cy) = \varphi(x + cy) = 0$

$$\psi(s) = ne^{-\sqrt{n}} \sin(ns)$$

$$c = i$$

$$\therefore u_n(x,y) = \frac{1}{2i}\int_{x-iy}^{x+iy} ne^{-\sqrt{n}} \sin(ns)\, ds$$

$$= \frac{ne^{-\sqrt{n}}}{2i} \int_{x-iy}^{x+iy} sin(ns)\, ds$$

$$u_n(x,y) = \frac{ne^{-\sqrt{n}}}{2i}\left[\frac{-cos(ns)}{n}\bigg|_{x-iy}^{x+iy}\right]$$

$$= \frac{-ne^{-\sqrt{n}}}{2in}[cosn(x+iy) - cosn(x+iy)]$$

$$u_n(x,y) = \frac{-e^{-\sqrt{n}}}{2i}\ [cos(nx+iny) - cos(nx-iny)]$$

$$= \frac{-e^{-\sqrt{n}}}{2i}\ [cos(nx)cos(iny) - sin(nx)sin(iny) - (cos(nx)cos(iny) + sin(nx)sin(iny))]$$

$$= \frac{-e^{-\sqrt{n}}}{2i}\ [cos(nx)cos(iny) - sin(nx)sin(iny) - cos(nx)cos(iny) - sin(nx)sin(iny)]$$

$$= \frac{-e^{-\sqrt{n}}}{2i}\ [-2sin(nx)sin(iny)]$$

$$= \frac{e^{-\sqrt{n}}}{i}\ [sin(nx)sin(iny)]$$

$$\therefore\ u_n(x,y) = \frac{e^{-\sqrt{n}}}{i}\ sin(nx)\ i\ sinh(ny)$$

$$= e^{-\sqrt{n}}\ sin(nx)sinh(ny)$$

$$\therefore\ u_n(x,y) = e^{-\sqrt{n}}\ sin(nx)\frac{e^{ny}-e^{-ny}}{2}$$

$$\therefore\ u_n(x,y) = \frac{e^{-\sqrt{n}}}{2}\ (e^{ny} - e^{-ny})sin(nx)\ \textit{is the solution}$$

$$\textit{not}\ \therefore\ u_n(x,y) = e^{-\sqrt{n}}\ sin(nx)e^{ny}\ \ \textit{as stated in the question}$$

Verification

Claim

$$u_{xx} + u_{yy} = 0$$

Proof $$u_x = \frac{ne^{-\sqrt{n}}}{2}(e^{ny} - e^{-ny})\cos(nx)$$

$$u_{xx} = -\frac{n^2e^{-\sqrt{n}}}{2}(e^{ny} - e^{-ny})\sin(nx)$$

$$= -n^2u_n(x, y)$$

$$u_y = \frac{e^{-\sqrt{n}}}{2}(ne^{ny} + ne^{-ny})\sin(nx)$$

$$u_{yy} = \frac{e^{-\sqrt{n}}}{2}\left(n^2e^{ny} + n^2e^{-ny}\right)\sin(nx)$$

$$= n^2u_n(x, y)$$

$$\therefore u_{xx} + u_{yy} = -n^2u_n(x, y) + n^2u_n(x, y) = 0$$

$$\therefore \nabla^2u(x, y) = 0 \;\; \forall x \text{ and } t > 0$$

$$u(x, 0) = \frac{e^{-\sqrt{n}}}{2}(1 - 1)\sin(nx) = 0$$

$$u_y(x, y) = \frac{e^{-\sqrt{n}}}{2}(ne^{ny} + ne^{-ny})\sin(nx)$$

$$= \frac{ne^{-\sqrt{n}}}{2}(e^{ny} + e^{-ny})\sin(nx)$$

$$\therefore u_y(x, 0) = \frac{ne^{-\sqrt{n}}}{2}(1 + 1)\sin(nx)$$

$$\therefore u_y(x, 0) = ne^{-\sqrt{n}}\sin(nx)$$

Hence, $u_n(x, y) = \frac{e^{-\sqrt{n}}}{2}(e^{ny} + e^{-ny})\sin(nx)$ ***is a solution.***

Note $u_n(x, y) = e^{-\sqrt{n}}\sin(nx)e^{ny}$ ***does not satisfy*** $u(x, 0) = 0$.

In particular $u(x,0) = e^{-\sqrt{n}}\,sin(nx) \neq 0.$

Hence, cannot be a solution to the cauchy problem.

Second Part

ie to prove that for any positive $y_0, u_n(x,y_0) \to \infty$ *as* $n \to \infty$

$$u_n(x,y_0) = \frac{e^{-\sqrt{n}}}{2}(e^{ny_0} + e^{-ny_0})sin(nx)$$

$$\lim_{n\to\infty} u_n(x,y_0) = \lim_{n\to\infty}\left(\frac{e^{-\sqrt{n}}}{2}(e^{ny_0} + e^{-ny_0})sin(nx)\right).$$

By the properties of limit, we have

$$\lim_{n\to\infty} u_n(x,y_0) = \frac{1}{2}\lim_{n\to\infty}(e^{-\sqrt{n}}(e^{ny_0} + e^{-ny_0}))\lim_{n\to\infty} sin(nx)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(e^{-\sqrt{n}}(e^{ny_0} + e^{-ny_0})\right)\lim_{n\to\infty}\left(\frac{e^{inx}-e^{-inx}}{2i}\right).$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(e^{ny_0-\sqrt{n}} + e^{-(ny_0+\sqrt{n})}\right)\lim_{n\to\infty}\left(\frac{e^{inx}-e^{-inx}}{2i}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(e^{n(y_0-\frac{1}{\sqrt{n}})} + e^{-n(y_0+\frac{1}{\sqrt{n}})}\right)\frac{1}{2i}\lim_{n\to\infty}(e^{inx} - e^{-inx})$$

By the properties of limit, we have

$$\lim_{n\to\infty} u_n(x,y_0) = \frac{1}{2}\left\{\lim_{n\to\infty} e^{n\left(y_0-\frac{1}{\sqrt{n}}\right)} + \lim_{n\to\infty} e^{-n\left(y_0+\frac{1}{\sqrt{n}}\right)}\right\}$$

$$\frac{1}{2i}[\lim_{n\to\infty} e^{inx} - \lim_{n\to\infty} e^{-inx}] **$$

***Note that the following*:**

$$1]\ \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$$

$$2]\lim_{n\to\infty} n = \infty$$

$$3]\lim_{n\to\infty} e^{inx} = \infty$$

$$4]\lim_{n\to\infty} e^{-inx} = 0$$

$$5]\lim_{n\to\infty} (e^{-n\left(y_0+\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{1}{e^{n\left(y_0+\frac{1}{\sqrt{n}}\right)}} = 0$$

$$6]\ \lim_{n\to\infty} e^{n\left(y_0-\frac{1}{\sqrt{n}}\right)} = \infty, \textit{since } y_0 > 0 \textit{ and } \frac{1}{\sqrt{n}} \to 0 \textit{ and } n \to \infty$$

$Hence\ from ** \ we\ have,$

$$\lim_{n\to\infty} u_n(x, y_0) = \infty + 0 + \infty - 0$$

$$\lim_{n\to\infty} u_n(x, y_0) = \infty \textit{ as desired}$$