Section 4.8

In each of the problems below, write the solution of the intial – boundary value problem on [0, L] for the given information, using the Fourier method.

$$1)\ \varphi(x) = 1 - \cos(x), \psi(x) = 0, c = 3, L = 2\pi$$

$$2)\ \varphi(x) = 0, = \psi(x) = e^{-x}, c = 3, L = 2$$

Problem 1

$$\varphi(x) = 1 - \cos(x), \psi(x) = 0, c = 3, L = 2\pi$$

Solution

$$a_n = \frac{2}{L}\int_0^L \varphi(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$

$$= \frac{2}{2\pi}\int_0^{2\pi}(1 - \cos(\xi))\sin\left(\frac{n\pi\xi}{2\pi}\right)d\xi$$

$$= \frac{1}{\pi}\int_0^{2\pi}(1 - \cos(\xi))\sin\left(\frac{n\xi}{2}\right)d\xi$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{n\xi}{2}\right)d\xi - \frac{1}{\pi}\int_0^{2\pi}\cos(\xi)\sin\left(\frac{n\xi}{2}\right)d\xi$$

$$\text{since } \cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{n\xi}{2}\right)d\xi - \frac{1}{2\pi}\int_0^{2\pi}\left[\sin\left(\xi + \frac{n\xi}{2}\right) - \sin\left(\xi - \frac{n\xi}{2}\right)\right]d\xi$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{n\xi}{2}\right)d\xi - \frac{1}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{2+n}{2}\right)\xi - \sin\left(\frac{2-n}{2}\right)\xi\right]d\xi$$

$$\text{for } n = 2$$

$$a_2 = \frac{1}{\pi}\int_0^{2\pi}\sin(\xi)d\xi - \frac{1}{2\pi}\int_0^{2\pi}[\sin(2\xi) - \sin(0)\xi]d\xi$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sin(\xi)d\xi - \frac{1}{2\pi}\int_0^{2\pi}\sin(2\xi)d\xi$$

$$= \frac{1}{\pi}\left[-cos(\xi) + \frac{1}{2\pi}\left(\frac{cos(2\xi)}{2}\right)\right] |_{0}^{2\pi}$$

$$= \frac{1}{\pi}\left[-cos(\xi) + \frac{1}{4}\left(cos(2\xi)\right)\right] |_{0}^{2\pi}$$

$$= \frac{1}{\pi}\left[-\left(cos2\pi - cos(0)\right) + \frac{1}{4}\left(cos4\pi - cos(0)\right)\right] = 0$$

$$\therefore a_2 = 0$$

$for\ n = 1, 3, 4 \ldots.$

$$= \frac{1}{\pi}\left[\frac{-cos\left(\frac{n\xi}{2}\right)}{\frac{n}{2}} - \frac{1}{2}\left(\frac{-cos\left(\frac{2+n}{2}\right)\xi}{\frac{2+n}{2}} + \frac{cos\left(\frac{2-n}{2}\right)\xi}{\frac{2-n}{2}}\right)\right] |_{0}^{2\pi}$$

$$= \frac{1}{\pi}[\frac{-2}{n}(cosn\pi - cos0) - \frac{1}{2}(\frac{-2}{n+2}cos\left(\frac{2+n}{2}\right)2\pi - cos0$$

$$+ \frac{2}{2-n}\left(cos\left(\frac{2-n}{2}\right)2\pi - cos0\right))]$$

$$= \frac{1}{\pi}[\frac{-2}{n}(cosn\pi - 1) + \frac{1}{(n+2)}\left(cos\left(\frac{2+n}{2}\right)2\pi - 1\right)$$

$$- \frac{1}{(2-n)}\left(cos\left(\frac{2-n}{2}\right)2\pi - 1\right)]$$

$$= \frac{1}{\pi}[\frac{-2}{n}cosn\pi + \frac{2}{n} + \frac{1}{(n+2)}\left(cos\left(1 + \frac{n}{2}\right)2\pi\right)$$

$$- \frac{1}{(n+2)} - \frac{1}{(2-n)}\left(cos\left(1 - \frac{n}{2}\right)2\pi + \frac{1}{(2-n)}\right)]$$

$$= \frac{1}{\pi}[\frac{-2}{n}cosn\pi + \frac{1}{(n+2)}\left(cos\left(1 + \frac{n}{2}\right)2\pi\right) - \frac{1}{(2-n)}cos\left(1 - \frac{n}{2}\right)2\pi$$

$$+ \frac{2}{n} - \frac{1}{(n+2)} + \frac{1}{(2-n)}]$$

$$= \frac{1}{\pi}[\frac{-2}{n}cosn\pi + \frac{1}{(n+2)}(cos(2\pi + n\pi)) - \frac{1}{(2-n)}cos(2\pi - n\pi)$$

$$+ \frac{2}{n} - \frac{1}{(n+2)} + \frac{1}{(2-n)}]$$

$$=\frac{1}{\pi}[\frac{-2}{n}cosn\pi+\frac{1}{(n+2)}(cos2\pi cosn\pi-sin2\pi sinn\pi$$

$$-\frac{1}{(2-n)}(cos2\pi cosn\pi+sin2\pi sinn\pi)+\frac{2}{n}-\frac{1}{(n+2)}+\frac{1}{(2-n)}]$$

$$=\frac{1}{\pi}\left[\frac{-2}{n}cos(n\pi)+\frac{1}{(n+2)}cos(n\pi)-\frac{1}{(2-n)}cos(n\pi)+\frac{2}{n}-\frac{1}{(n+2)}+\frac{1}{(2-n)}\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-2}{n}+\frac{1}{(2+n)}-\frac{1}{(2-n)}\right)cos(n\pi)+\frac{2}{n}-\frac{1}{(2+n)}+\frac{1}{(2-n)}\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-2}{n}+\frac{1}{(2+n)}-\frac{1}{(2-n)}\right)(cos(n\pi)-1)\ \right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-2(4-n^2)+n(2-n)-n(2+n)}{n(4-n^2)}\right)\left(cos^2\left(\frac{n\pi}{2}\right)-sin^2\left(\frac{n\pi}{2}\right)-1\right)\ \right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-8+2n^2+2n-n^2-2n-n^2}{n(4-n^2)}\right)\left(-2sin^2\left(\frac{n\pi}{2}\right)\right)\ \right]$$

$$=\frac{1}{\pi}\left[\left(\frac{-8}{n(4-n^2)}\right)\left(-2sin^2\left(\frac{n\pi}{2}\right)\right)\ \right]$$

$$=\frac{16}{\pi n(4-n^2)}\ sin^2\left(\frac{n\pi}{2}\right)$$

$$\therefore a_n=\ \frac{16}{\pi n(4-n^2)}\ sin^2\left(\frac{n\pi}{2}\right)$$

$$since\ \psi(x)=0\Rightarrow\ b_n=0$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}\left(a_n cos\left(\frac{n\pi ct}{L}\right)+b_n sin\left(\frac{n\pi ct}{L}\right)\right)sin\ \left(\frac{n\pi x}{L}\right)$$

$$=\sum_{n=1,3,\dots}^{\infty}[\frac{16}{\pi n(4-n^2)}\ sin^2\left(\frac{n\pi}{2}\right)cos\left(\frac{3nt}{2}\right)sin\ \left(\frac{nx}{2}\right)],n\neq 2$$

Problem 5

$$\varphi(x) = 0, = \psi(x) = e^{-x} \; c = 3, L = 2$$

Solution

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{where } a_n = \frac{2}{L}\int_0^L \varphi(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi \text{ and}$$

$$b_n = \frac{2}{n\pi c}\int_0^L \psi(\xi)\sin\left(\frac{n\pi\xi}{c}\right)d\xi$$

$$\text{since } \varphi(x) = 0 \;\Rightarrow a_n = 0$$

$$b_n = \frac{2}{3n\pi}\int_0^2 e^{-\xi}\sin\left(\frac{n\pi\xi}{2}\right)d\xi$$

$$\text{integrate } \int e^{-\xi}\sin\left(\frac{n\pi\xi}{2}\right)d\xi$$

using integration by parts

$$u = e^{-\xi}, \quad du = -e^{-\xi}d\xi$$

$$dv = \sin\left(\frac{n\pi\xi}{2}\right)d\xi, \; v = \frac{-\cos\left(\frac{n\pi\xi}{2}\right)}{\frac{n\pi}{2}} = -\frac{2}{n\pi}\cos\left(\frac{n\pi\xi}{2}\right)$$

$$\int e^{-\xi}\sin\left(\frac{n\pi\xi}{2}\right)d\xi = \left\{\frac{-2e^{-\xi}\cos\left(\frac{n\pi\xi}{2}\right)}{n\pi} - \frac{2}{n\pi}\int e^{-\xi}\cos\left(\frac{n\pi\xi}{2}\right)d\xi\right\}$$

using integration by parts for the second part

$$u = e^{-\xi}, \quad du = -e^{-\xi}d\xi$$

$$dv = \cos\left(\frac{n\pi\xi}{2}\right)d\xi, \; v = \frac{\sin\left(\frac{n\pi\xi}{2}\right)}{\frac{n\pi}{2}} = \frac{2}{n\pi}\sin\left(\frac{n\pi\xi}{2}\right)$$

$$= \left\{\frac{-2e^{-\xi}\cos\left(\frac{n\pi\xi}{2}\right)}{n\pi} - \frac{2}{n\pi}\left[-\frac{2}{n\pi}e^{-\xi}\sin\left(\frac{n\pi\xi}{2}\right) + \frac{2}{n\pi}\int e^{-\xi}\sin\left(\frac{n\pi\xi}{2}\right)d\xi\right]\right\}$$

$$= \left\{\frac{-2}{n\pi} e^{-\xi} cos\left(\frac{n\pi\xi}{2}\right) + \frac{4}{(n\pi)^2} e^{-\xi} sin\left(\frac{n\pi\xi}{2}\right) - \frac{4}{(n\pi)^2}\int e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi\right\}$$

$$\int e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi = \{\frac{-2}{n\pi} e^{-\xi} cos\left(\frac{n\pi\xi}{2}\right) + \frac{4}{(n\pi)^2} e^{-\xi} sin\left(\frac{n\pi\xi}{2}\right)$$

$$- \frac{4}{(n\pi)^2}\int e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi\}$$

$$1 + \frac{4}{(n\pi)^2}\ \int e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi = \frac{-2}{n\pi} e^{-\xi} cos\left(\frac{n\pi\xi}{2}\right) + \frac{4}{(n\pi)^2} e^{-\xi} sin\left(\frac{n\pi\xi}{2}\right)$$

$$\therefore\ \ \int e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi = \frac{(n\pi)^2}{(n\pi)^2+4}\left[\frac{-2}{n\pi} e^{-\xi} cos\left(\frac{n\pi\xi}{2}\right) + \frac{4}{(n\pi)^2} e^{-\xi} sin\left(\frac{n\pi\xi}{2}\right)\right]$$

$$\therefore \int_0^2 e^{-\xi}\, sin\left(\frac{n\pi\xi}{2}\right) d\xi = \frac{(n\pi)^2}{(n\pi)^2+4}\frac{2}{n\pi}[-(e^{-2} cos(n\pi) - cos0)$$

$$\frac{2}{n\pi}\left(e^{-\xi} sin(n\pi) - sin0\right)]$$

$$= \frac{2(n\pi)}{(n\pi)^2+4}[-(e^{-2} cos(n\pi) + 1]$$

$$= \frac{2(n\pi)}{(n\pi)^2+4}[1 - e^{-2} cos(n\pi)]$$

$$\therefore b_n = \frac{2}{3n\pi}\left(\frac{2(n\pi)}{(n\pi)^2+4}\right)[1 - e^{-2} cos(n\pi)]$$

$$= \frac{4}{3(n\pi)^2+4}[1 - e^{-2} cos(n\pi)]$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{4[1-e^{-2} cos(n\pi)]}{3(4+n^2\pi^2)}\ sin\left(\frac{3n\pi t}{2}\right) sin\left(\frac{n\pi x}{2}\right)$$

Problem 14

Use separation of variables to solve the telegraph equation

$$u_{tt} + Au_t + Bu = c^2 u_{xx} \text{ for } 0 < x < L, t > 0$$

in which $A, B,$ and c are positive constants.

The boundary conditions are

$$u(0,t) = u(L,t) = 0, t > 0$$

and the intial conditions are

$$u(x,0) = \varphi(x) \text{ and } u_t(x,0) = 0, 0 < x < L$$

Assume that $A^2L^2 < 4(BL^2 + c^2\pi^2)$.

Solution

let $u(x,t) = X(x)T(t)$

$$\therefore u_{tt} + Au_t + Bu = XT'' + AXT' + BXT = c^2X''T$$

Divide through by XT

$$\Rightarrow \frac{T''}{T} + \frac{AT'}{T} + B = c^2\frac{X''}{X}$$

Since the left side is afunction of T but the right side is a function of X. This is only possible if they are constants.

$$\frac{T''}{T} + \frac{AT'}{T} + B = c^2\frac{X''}{X} = -\lambda,$$

where λ is a separation constant.

Now we have

$$\frac{T''}{T} + \frac{AT'}{T} + B = -\lambda \text{ ……… } i$$

and

$$c^2 \frac{X''}{X} = -\lambda \ldots\ldots\ldots ii$$

from ii we have

$$c^2 X'' + \lambda X = 0$$

Apply the boundary conditions, we have

$$u(0,t) = X(0)T(t) = 0 \quad \forall t \Rightarrow x(0) = 0$$

similarly,

$$u(L,t) = X(L)T(t) = 0 \quad \forall t \Rightarrow x(L) = 0$$

Note: $T(t) = 0 \quad \forall t \Rightarrow \ u(x,t) = 0$

which is also a solution if the initial displacement

and velocity are identically zero.

Now the problem becomes

$$c^2 X'' + \lambda X = 0,\ X(0) = 0,\ X(L) = 0$$

<u>**3 *cases***</u>

<u>***case* 1**</u>

$$\lambda = 0 \ \Rightarrow\ c^2 X'' = 0 \ \Rightarrow c^2 X' = a \ \Rightarrow\ \ c^2 X = ax + b$$

where $a, b =$ *constants*

apply $X(0) = 0 \ \Rightarrow\ \ 0 = b$

$$\therefore\ \ b = 0 \ \Rightarrow\ \ c^2 X = ax$$

apply $X(L) = 0 \ \Rightarrow\ \ \ 0 = aL \ \Rightarrow\ \ a = 0$

$\therefore$ *no non trivial solution for* $\lambda = 0$

case 2

$\lambda < 0 \quad say \ \lambda = -k^2$

$\therefore \ c^2 X'' - k^2 X = 0$

$let \ X = e^{mx}$

$\therefore \ c^2 \, m^2 \, e^{mx} + k^2 e^{mx} = 0$

$(c^2 \, m^2 - k^2) e^{mx} = 0$

$since \ e^{mx} \neq 0 \ \Rightarrow \ c^2 \, m^2 = k^2$

$\therefore m = \pm \frac{k}{c}$

$Hence, \ X = a e^{\frac{k}{c}x} + b e^{\frac{-k}{c}x}$

$X(0) = 0 \quad \Rightarrow 0 = a + b = b = -a$

$X(L) = 0 \ \Rightarrow 0 = \ a e^{\frac{kL}{c}} + b e^{\frac{-kL}{c}}$

$\therefore \ a e^{\frac{kL}{c}} - \ a e^{\frac{-kL}{c}} = 0 = a\left(e^{\frac{kL}{c}} - a e^{\frac{-kL}{c}}\right) = 0$

$since \ e^{\frac{kL}{c}} - \ e^{\frac{-kL}{c}} \neq 0 \ unless \ K \ or \ L = 0 \ which \ is \ not.$

$\therefore a = 0. \ \ Thus \ b = 0$

same trivial soln.

Case 3

$\lambda > 0 \quad say \ \lambda = k^2$

$\therefore \ c^2 X'' + k^2 X = 0$

$let \ X = e^{mx}$

$$\therefore\ c^2 m^2 e^{mx} + k^2 e^{mx} = 0$$

$$(c^2 m^2 + k^2)e^{mx} = 0$$

$$since\ e^{mx} \neq 0 \ \Rightarrow\ c^2 m^2 + k^2 = 0$$

$$\therefore m = \pm i\frac{k}{c}$$

$$Hence, X = a\cos\left(\frac{k}{c}x\right) + b\sin\left(\frac{k}{c}x\right)$$

$$X(0) = 0 \quad \Rightarrow 0 = a \quad \therefore a = 0$$

$$\therefore X = b\sin\left(\frac{k}{c}x\right) \Rightarrow$$

$$X(L) = 0 \ \Rightarrow\ 0 = b\sin\left(\frac{kL}{c}\right)$$

$$Since\ we\ need\ non\ trivial\ soln\ \Rightarrow \sin\left(\frac{kL}{c}\right) = 0$$

$$i.e\ \left(\frac{kL}{c}\right) = n\pi \ \Rightarrow k = \frac{cn\pi}{L} \quad \Rightarrow k^2 = \frac{c^2\,\pi^2\,n^2}{L^2}$$

$$\therefore\ \lambda_n = \frac{c^2\,\pi^2\,n^2}{L^2}\ for\ n = 1, 2, \dots\dots$$

$$X_n(x) = \sin\left(\frac{\pi n x}{L}\right)$$

$$\therefore i\ \ becomes\ \frac{T''}{T} + A\frac{T'}{T} + B = -\frac{c^2\,\pi^2\,n^2}{L^2}$$

$$T'' + AT' + \left(B + \frac{c^2\,\pi^2\,n^2}{L^2}\right)T = 0$$

$$let\ T(t) = e^{mt}$$

$$m^2 e^{mt} + Ame^{mt} + \left(B + \frac{c^2\,\pi^2\,n^2}{L^2}\right)e^{mt} = 0$$

$$\left(m^2 + Am + \left(B + \frac{c^2\,\pi^2\,n^2}{L^2}\right)\right)e^{mt} = 0$$

$\boldsymbol{since\ e^{mt} \neq 0}$

$$\therefore m^2 + Am + \left(\frac{BL^2 + c^2\pi^2 n^2}{L^2}\right) = 0$$

$$\therefore m = \frac{-A \pm \sqrt{A^2 - \frac{4(BL^2 + c^2\pi^2 n^2)}{L^2}}}{2}$$

$$= \frac{\frac{-A \pm \sqrt{A^2L^2 - 4(BL^2 + c^2\pi^2 n^2)}}{L}}{2}$$

$$= \frac{-AL \pm \sqrt{A^2L^2 - 4(BL^2 + c^2\pi^2 n^2)}}{2L}$$

$Since\ \ A^2L^2 < \ 4(BL^2 + c^2\pi^2)$

$\therefore A^2L^2 < \ 4(BL^2 + c^2\pi^2 n^2), since\ n\ positive\ real\ number$

$Thus,\ \ A^2L^2 - 4(BL^2 + c^2\pi^2 n^2) < 0$

$say\ \ \ A^2L^2 - \ 4(BL^2 + c^2\pi^2 n^2) = -\eta^2$

$$\therefore m = \frac{-AL \pm \sqrt{-\eta^2}}{2L} = \frac{-AL \pm i\eta}{2L}$$

$$\Rightarrow m = \frac{-A}{2} \pm \frac{i\eta}{2L}$$

$$\therefore T = e^{\frac{-At}{2}}\left[a\cos\left(\frac{\eta}{2L}\right) + b\sin\left(\frac{\eta}{2L}\right)\right]$$

$$\therefore T_n(t) = e^{\frac{-At}{2}}\left[a_n\cos\left(\frac{\eta}{2L}\right) + b_n\sin\left(\frac{\eta}{2L}\right)\right]$$

$where\ \ \eta = \sqrt{4(BL^2 + c^2\pi^2 n^2) - A^2L^2}$

$$\therefore u(x,t) = X_n(x)T_n(t)$$

$$= e^{\frac{-At}{2}}\left[a_n\cos\left(\frac{\eta t}{2L}\right) + b_n\sin\left(\frac{\eta t}{2L}\right)\right]\sin\ \left(\frac{\pi n x}{L}\right)$$

$We\ need\ this\ solution\ to\ satisfy\ the\ initial\ conditions\ i.e$

$u(x,0) = \varphi(x)\ \ and\ \ u_t(x,0) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{-At}{2}} \left\{a_n cos\left(\frac{\eta t}{2L}\right) + b_n sin\left(\frac{\eta t}{2L}\right)\right\} sin\ \left(\frac{\pi nx}{L}\right)$$

$$\therefore\ u_n\ (x,0) = \sum_{n=1}^{\infty} e^0\ \{a_n cos(0) + b_n sin(0)\} sin\ \left(\frac{\pi nx}{L}\right) = \varphi(x)$$

$$= \sum_{n=1}^{\infty} a_n\ sin\ \left(\frac{\pi nx}{L}\right) = \varphi(x)$$

There is a Fourier sine expansion on $[0, L]$

$$\therefore a_n = \frac{2}{L}\int_0^L \varphi(\xi) sin\ \left(\frac{\pi nx}{L}\right) d\xi$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\Big[\frac{-A}{2} e^{\frac{-At}{2}} \left\{a_n cos\left(\frac{\eta t}{2L}\right) + b_n sin\left(\frac{\eta t}{2L}\right)\right\}$$

$$+e^{\frac{-At}{2}} \left\{-a_n \frac{\eta}{2L} sin\left(\frac{\eta t}{2L}\right) + b_n \frac{\eta}{2L} cos\left(\frac{\eta t}{2L}\right)\right\}\Big] sin\ \left(\frac{\pi nx}{L}\right)$$

$$= \sum_{n=1}^{n} e^{\frac{-At}{2}} \Big[\frac{-A}{2} \left\{a_n cos\left(\frac{\eta t}{2L}\right) + b_n sin\left(\frac{\eta t}{2L}\right)\right\}$$

$$+\frac{\eta}{2L}\left\{-a_n sin\left(\frac{\eta t}{2L}\right) + b_n cos\left(\frac{\eta t}{2L}\right)\right\}\Big] sin\ \left(\frac{\pi nx}{L}\right)$$

$$\therefore u_t(x,0) = \sum_{n=1}^{\infty}\Big[\frac{-A}{2}(a_n) + \frac{\eta}{2L} b_n sin\Big]\ \left(\frac{\pi nx}{L}\right) = 0 \ldots\ldots\ \ iii$$

$$= \sum_{n=1}^{\infty} \frac{-A}{2} a_n sin\left(\frac{\pi nx}{L}\right) + \sum_{n=1}^{\infty} \frac{\eta}{2L} b_n sin\ \left(\frac{\pi nx}{L}\right) = 0$$

$$= \sum_{n=1}^{\infty} \frac{-A}{2} a_n sin\left(\frac{\pi nx}{L}\right) = \sum_{n=1}^{\infty} \frac{\eta}{2L} b_n sin\ \left(\frac{\pi nx}{L}\right)$$

Since $\sum_{n=1}^{\infty} a_n sin\left(\frac{\pi nx}{L}\right) = \varphi(x)$

$$\therefore \frac{A}{2}\ \varphi(x) = \sum_{n=1}^{\infty} \frac{\eta}{2L} b_n sin\ \left(\frac{\pi nx}{L}\right)$$

$$\therefore\ \varphi(x) = \sum_{n=1}^{\infty} \frac{\eta}{AL} b_n sin\ \left(\frac{\pi nx}{L}\right)$$

This is the Fourier sine expansion of φ ***on*** $[0, L]$

$$\therefore \frac{n}{AL} b_n = \frac{2}{L}\int_0^L \varphi(\xi) sin\ \left(\frac{\pi nx}{L}\right) d\xi$$

$$\therefore b_n = \frac{2A}{\eta}\int_0^L \varphi(\xi) \sin\left(\frac{\pi n x}{L}\right) d\xi$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{\frac{-At}{2}} \left\{a_n \cos\left(\frac{\eta t}{2L}\right) + b_n \sin\left(\frac{\eta t}{2L}\right) \sin\left(\frac{\pi n x}{L}\right)\right\}$$

where

$$a_n = \frac{2}{L}\int_0^L \varphi(\xi) \sin\left(\frac{\pi n x}{L}\right) d\xi \text{ and}$$

$$b_n = \frac{2A}{\eta}\int_0^L \varphi(\xi) \sin\left(\frac{\pi n x}{L}\right) d\xi$$

with $\eta = \sqrt{4(BL^2 + c^2\,\pi^2\,n^2) - A^2L^2}$

A, B, C and L are constants.

Problem 15

Continue with the telegraph equation

$$u_{tt} + Au_t + Bu = u_{xx},\ 0 < x < L, t \geq 0$$

with $c = 1$. ***Suppose that***

$$u(0,t) = u(L,t) = 0 \ for \ t \geq 0$$

$$u(x,0) = \varphi(x),\ \ u_t(x,0) = 0 \ for \ 0 \leq x \leq L$$

Prove that for any $T > 0$,

$$\int_0^L (u_x^2 + u_t^2 + bu^2)_{t=T} dx \ \leq \ \int_0^L (u_x^2 + u_t^2 + bu^2)_{t=0} dx.$$

Hint: First show that $(2u_t u_x)_x = (u_x^2 + u_t^2 + bu^2)_t + \ 2au_t^2$

Solution

Same with problem 14 with $c = 1$

$$\therefore K = \frac{\pi n}{L} \ \therefore \ \lambda_n = \frac{\pi^2 n^2}{L^2},\ n = 1, 2, \dots \dots$$

$$X_n(x) = \sin\left(\frac{\pi n x}{L}\right)$$

Similarly, $T_n(t) = e^{\frac{-At}{2}} \left\{a_n \cos\left(\frac{\eta}{2L}\right) + b_n \sin\left(\frac{\eta}{2L}\right)\right\}$

where $\eta = \sqrt{4(BL^2 + c^2 \pi^2 n^2) - A^2 L^2}$

$$u(x,t) = X(t)T(t) = \ \sum_{n=1}^{\infty} e^{\frac{-At}{2}} \left\{a_n \cos\left(\frac{\eta t}{2L}\right) + b_n \sin\left(\frac{\eta t}{2L}\right)\right\} \sin\ \left(\frac{\pi n x}{L}\right)$$

Now for the initial condition $u_t(x,0) = \psi(x)$

we have from iii

$$u_t(x,0) = \sum_{n=1}^{\infty} \left\{\left(\frac{-A}{2} a_n + \frac{\eta}{2L} b_n\right) \sin\ \left(\frac{\pi n x}{L}\right)\right\} = \psi(x)$$

$$= \sum_{n=1}^{\infty} \frac{-A}{2} a_n \sin\left(\frac{\pi n x}{L}\right) + \sum_{n=1}^{\infty} \frac{\eta}{2L} b_n \sin\left(\frac{\pi n x}{L}\right) = \psi(x)$$

$$= \frac{-A}{2} \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right) + \frac{\eta}{2L} \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) = \psi(x)$$

since $\sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right) = \varphi(x)$

$$\Rightarrow \frac{-A}{2} \varphi(x) + \frac{\eta}{2L} \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) = \psi(x)$$

$$\Rightarrow \frac{A}{2} \varphi(x) + \psi(x) = \frac{\eta}{2L} \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right)$$

$$= \left[\frac{A}{2} \varphi(x) + \psi(x)\right] = \sum_{n=1}^{\infty} \frac{\eta}{2L} b_n \sin\left(\frac{\pi n x}{L}\right)$$

This is the fourier sine expansion of $\frac{A}{2} \varphi(x) + \psi(x)$ *on* $[0, L]$

$$\therefore \frac{\eta}{2L} b_n = \frac{2}{L} \int_0^L \left(\frac{A}{2} \varphi(\xi) + \psi(\xi)\right) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

$$\therefore b_n = \frac{4}{\eta} \int_0^L \left(\frac{A}{2} \varphi(\xi) + \psi(\xi)\right) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

Where $\varphi(\xi) = \sum_{n=1}^{\infty} \frac{\eta}{AL} b_n \sin\left(\frac{\pi n \xi}{L}\right)$.

I don't know how to continue from here because the initial condition is not homogenous like the previous problem.