Section 4.3

Problem 1

Suppose that the initial displacement and velocity functions vanish outside an interval of finite length.

Specifically, let a be a positive number and suppose that

$\varphi(x) = 0$ *and* $\psi(x) = 0$ *for* $|x| > a.$

Prove the the solution of the Cauchy problem

$u_{tt} = c^2 u_{xx}$ *for* $-\infty < x < \infty, t > 0$

$u(x,0) = \varphi(x), u_t(x,0) = \psi(x)$ *for* $-\infty < x < \infty$

vanishes outside $[-a - ct, a + ct]$

SOLUTION

We want to show that $u(x,t) = 0$ *if given* $a > 0$ *and*

$\varphi(x) = 0, \psi(x) = 0$ *for* $|x| > a$

i.e $u(x,t) = 0 \ \forall x \notin [-a - ct, a + ct]$

By d'Alembert's Formula, the solution for the problem is

$$u(x,t) = \frac{1}{2}\big(\varphi(x - ct) + \varphi(x + ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

If $t > 0$ *and* $x > a + ct$, *then,*

$x = a + ct + h$ *for some* $h > 0.$

Thus from d'Alembert formula we have

$\varphi(x - ct) = \varphi(a + ct + h - ct) = \varphi(a + h)$

since $a + h > a$

$\therefore\ \varphi(x-ct) = \ \varphi(a+h) = 0$

Similarly,

$\varphi(x+ct) = \varphi(a+ct+h+ct) = \varphi(a+2ct+h)$

$since\ a+2ct+h > a$

$\therefore\ \varphi(x+ct) = \varphi(a+2ct+h) = 0$

$\therefore u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds = \frac{1}{2c}\int_{a+ct+h-ct}^{a+ct+h+ct}\psi(s)ds$

$Since\ \psi(s) = 0, \forall\ s \in [a+h, a+2ct+h]$

$\therefore u(x,t) = 0\ for\ x > a+ct$

$Similarly, if\ t > 0\ \ and\ x < -a-ct,$

$then\ x = -a-ct-h\ for\ some\ h > 0$

$\therefore using\ the\ d'Alembert formula$

$u(x,t) = \ \frac{1}{2}\big(\varphi(x-ct)+\varphi(x+t)\big) + \ \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$

we have,

$\varphi(x-ct) = \varphi(-a-ct-h-ct) = \ \varphi(-a-2ct-h)$

$since\ -a-2ct-h < -a$

$\therefore\ \varphi(x-ct) = \ \varphi(-a-2ct-h) = 0$

Similarly,

$\varphi(x+ct) = \varphi(-a-ct-h+ct) = \varphi(-a-h)$

$since\ -a-h < -a$

$\therefore \varphi(x+ct) = \varphi(-a-h) = 0$

$$\therefore u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a-2ct-h}^{-a-h} \psi(s)ds$$

$Since\ \psi(s) = 0, \forall\ s \in [-a-2ct-h, -a-h]$

$\therefore u(x,t) = 0\ for\ x < -a-ct$

$\therefore The\ solution, u(x,t), of\ the\ problem\ vanishes$

$outside[-a-ct, a+ct]$