Section 3.4

Write the Fourier sine and the Fourier cosine series of the following functions

$1) f(x) = 4 \; for \; 0 \le x \le 3$

$3) \; f(x) = \begin{cases} 0 & for \; 0 \le x \le \frac{\pi}{2} \\ sinx & for \; \frac{\pi}{2} < x \le \pi \end{cases}$

$7) f(x) = sin(3x) \; for \; 0 \le x \le \pi$

Problem 1

$1) \; f(x) = 4 \; for \; 0 \le x \le 3$

Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} bn \; sin \left(\frac{n\pi x}{L}\right)$$

$$bn = \frac{2}{L} \int_0^L f(x) \, sin\left(\frac{n\pi x}{L}\right) dx$$

where $L = 3$ ***and*** $f(x) = 4$

$$\therefore \; bn = \frac{2}{3} \int_0^3 4 \, sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{8}{3} \int_0^3 sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{8}{3} \left[\frac{-cos\left(\frac{n\pi x}{3}\right)}{\frac{n\pi}{3}}\right] \Big|_0^3$$

$$= \frac{-8}{3} \times \frac{3}{n\pi} \; cos\left(\frac{n\pi x}{3}\right) \Big|_0^3$$

$$= \frac{-8}{n\pi} cos\left(\frac{n\pi x}{3}\right) \Big|_0^3$$

$$= \frac{-8}{n\pi} [cos(n\pi) - cos0]$$

$$= \frac{-8}{n\pi}[(-1)^n - 1]$$

$$= \frac{-8}{n\pi}[1 - (-1)^n]$$

$for\ n = even$

$bn = 0$

$for\ n = odd$

$i.e\ n = 2k - 1, k = 1, 2, 3 \ldots \ldots.$

$$bn = \frac{8}{(2k-1)\pi}(2) = \frac{16}{(2k-1)\pi}$$

$\therefore$ *The sine series for the function is*

$$\sum_{n=1}^{\infty} \frac{16}{(2k-1)\pi} sin\left(\frac{(2n-1)}{3}\ \pi x\right)$$

<u>Fourier Cosine series</u>

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_o = \frac{2}{L}\int_0^L f(x)dx = \frac{2}{3}\int_0^3 4dx$$

$$\frac{2}{3}\ \ 4x|_0^3 = \frac{8}{3}(3 - 0) = 8$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 4\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{8}{3}\left[\sin\left(\frac{n\pi x}{3}\right).\frac{3}{n\pi}\right]|_0^3$$

$$= \frac{8}{n\pi}[\sin(n\pi) - \sin 0] = 0$$

$\therefore$ *The cosine series for the function is* $\frac{8}{2} = 4.$

$Hence, the\ cosine\ series\ for\ the\ function\ is\ the\ function\ itself.$

$$3)\ f(x) = \begin{cases} 0 & for\ 0 \le x \le \frac{\pi}{2} \\ sinx & for\ \frac{\pi}{2} < x \le \pi \end{cases}$$

<u>***Fourier sine series***</u>

$$f(x) = \ f(x) = \sum_{n=1}^{\infty} bn\ sin\ nx$$

$$bn = \frac{2}{\pi} \int_0^{\pi} f(x)\ sin\ nxdx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\ sin\ nxdx + \int_{\frac{\pi}{2}}^{\pi} f(x)\ sin\ nxdx\right]$$

$$= \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} sinx\ sin\ nxdx\ = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} 2sinx\ sin\ nxdx$$

$$since\ 2sinAsinB = cos(nx - x) - cos(nx + x)$$

$$\therefore bn = \ \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi}(cos(n-1)x - cos(n+1)x)dx \ldots\ldots(1)$$

$$= \frac{1}{\pi}\left[\left(\frac{sin(n-1)x}{n-1} - \frac{sin(n+1)x}{n+1}\right)\Big|_{\frac{\pi}{2}}^{\pi}\right]\ \ , n > 1$$

$$= \frac{1}{\pi}\left[\frac{sin(n-1)\pi - sin(n-1)\frac{\pi}{2}}{n-1} - \frac{sin(n+1)\pi - sin(n+1)\frac{\pi}{2}}{n+1}\right]$$

$$n = 2, 3, 4 \ldots.$$

$$= \frac{1}{\pi}\left[\frac{-sin(n-1)\frac{\pi}{2}}{n-1} + \frac{sin(n+1)\frac{\pi}{2}}{n+1}\right]$$

$$since\ sin(n-1)\pi = sin(n+1)\pi = 0\ \forall\ n \in Z$$

$$= \frac{1}{\pi}\left[\frac{sin(n+1)\frac{\pi}{2}}{n+1} + \frac{sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$since\ sin(n+1)\frac{\pi}{2} = -\ sin(n-1)\frac{\pi}{2} = cos\frac{n\pi}{2}$$

$$\therefore \frac{1}{\pi}\left[\frac{cos\frac{n\pi}{2}}{n+1} - \frac{-cos\frac{n\pi}{2}}{n-1}\right] = \frac{1}{\pi}\left[\frac{cos\frac{n\pi}{2}}{n+1} + \frac{cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(n-1+n+1)cos\frac{n\pi}{2}}{n^2-1}\right] = \frac{1}{\pi}\left[\frac{2ncos\frac{n\pi}{2}}{n^2-1}\right]$$

$$\therefore f(x) = bsinx + \sum_{n-2}^{\infty} \frac{2ncos\frac{n\pi}{2}}{\pi(n^2-1)} sin\, nx$$

$for\ b_1$

$Consider\ from\ (1)$

$$b_n = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(cos(n-1) - cos(n-1)x)dx$$

$for\ n = 1$

$$b_1 = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(1 - cos2x)dx = \frac{1}{\pi}\left[x - \frac{sin2x}{2}\right]\Big|_{\frac{\pi}{2}}^{\pi}$$

$$\therefore b_1 = \frac{1}{\pi}\left[\left(\pi - \frac{\pi}{2}\right) - \left(\frac{sin2\pi - sin\pi}{2}\right)\right] = \frac{1}{\pi}\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$\therefore The\ sine\ series\ for\ the\ function\ is$

$$\frac{1}{2}sinx + \sum_{n=2}^{\infty} \frac{2ncos\frac{n\pi}{2}}{\pi(n^2-1)} sin\, nx$$

Fourier cosine series

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n\, cos\, nx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)dx$$

$$\therefore a_o = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} sinxdx = \frac{2}{\pi}(-cosx\left|_{\frac{\pi}{2}}^{\pi}\right|$$

$$= -\frac{2}{\pi}\left[cos\pi - cos\frac{\pi}{2}\right] = \frac{2}{\pi}$$

$$\therefore a_o = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)cos(nx)dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)cos(nx)dx$$

$$a_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} sinxcosxdx = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 2sinxcosxdx$$

$$since\ 2sinxcosnx = sin(n+1)x + sin(1-n)x$$

$$= sin(n+1)x - sin(n-1)x$$

$$a_n = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(sin(n+1) - sin(n-1)x)dx$$

$$for\ n = 1, we\ have$$

$$a_1 = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} sin2xdx = -\frac{1}{\pi}\frac{cos2x}{2}\left|\begin{matrix}\pi\\ \frac{\pi}{2}\end{matrix}\right|$$

$$= -\frac{1}{2\pi}[cos2\pi - cos\pi]$$

$$= -\frac{1}{2\pi}[1+1] = -\frac{1}{\pi}\ \therefore a_1 = \frac{-1}{\pi}$$

$$for\ n = 2, 3, \ldots\ldots\ldots, we\ have$$

$$a_n = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}(sin(n+1)x - sin(n-1)x)dx$$

$$= \frac{1}{\pi}\left[\frac{-cos(n+1)x}{n+1} + \frac{cos(n-1)x}{n-1}\right]\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-cos(n+1)\pi + cos(n+1)\frac{\pi}{2}}{n+1} + \frac{cos(n-1)\pi - cos(n-1)\frac{\pi}{2}}{n-1}\right]^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left[\frac{cos(n-1)\pi}{n-1} - \frac{cos(n+1)\pi}{n+1} + \frac{cos(n+1)\frac{\pi}{2}}{n+1} - \frac{cos(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$since\ cos(n-1)\pi = cos(n+1)\pi = -cos(n\pi)$$

$$and\ cos(n+1)\frac{\pi}{2} = -cos(n+1)\frac{\pi}{2} = sin(\frac{\pi}{2}n)$$

$$\therefore\ a_n = \frac{1}{\pi}\left[\frac{-\cos n\pi}{n-1} + \frac{\cos n\pi}{n+1} - \frac{\sin\frac{\pi}{2}n}{n+1} - \frac{\sin\frac{\pi}{2}n}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-(n+1)\cos n\pi + (n-1)\cos n\pi}{n^2-1}\right) - \left(\frac{(n-1)\sin\frac{\pi}{2}n + (n+1)\sin\frac{\pi}{2}n}{n^2-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{(n-1-n-1)\cos n\pi}{n^2-1}\right) - \left(\frac{(n-1+n+1)\sin\frac{\pi}{2}n}{n^2-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\cos n\pi}{n^2-1} - \frac{2n\sin\frac{\pi}{2}n}{n^2-1}\right]$$

$$= \frac{-2}{\pi}\left[\frac{\cos(n\pi) + n\sin(n\frac{\pi}{2})}{n^2-1}\right]$$

$\therefore$ ***The fourier cosine series for the function is:***

$\frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos(nx)$ ***which is:***

$$= \frac{1}{\pi} - \frac{\cos x}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\cos(n\pi) + n\sin(n\frac{\pi}{2})}{n^2-1}\cos nx$$

Problem 7

$f(x) = \sin(3x)$ *for* $0 \le x \le \pi$

Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} bn\,\sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin 3x\,\sin(nx)\,dx == \frac{1}{\pi}\int_0^{\pi} 2\sin 3x\,\sin(nx)\,dx$$

$$\text{since } 2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$\therefore\ bn = \frac{1}{\pi}\int_0^{\pi}\big(\cos(3x-nx) - \cos(3x+nx)\big)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\big(\cos(nx-3x)-\cos(nx+3x)\big)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\cos(n-3)x-\cos(n+3)x)\,dx$$

$for\ n = 3, we\ have$

$$b_3 = \frac{1}{\pi}\int_0^{\pi}(\cos 0-\cos 6x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(1-\cos 6x)dx = \frac{1}{\pi}\left[x-\frac{\sin 6x}{6}\right]\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi-\frac{(\sin 6x-\sin 0)}{6}\right] = 1$$

$\therefore b_3 = 1$

$for\ n = 1, 2, 4, 5 \ldots\ldots$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\cos(n-3)x-\cos(n+3)x)dx$$

$$b_n = \frac{1}{\pi}\left[\frac{\sin(n-3)x}{n-3}-\frac{\sin(n+3)x}{n+3}\right]\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-3)\pi-\sin 0}{n-3}-\frac{\sin(n+3)\pi-\sin 0}{n+3}\right] = 0$$

$\therefore$ *The fourier sine series is*

$f(x) = \sin 3x$ *i.e the function itself.*

Fourier cosine series

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\ nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin 3x\ dx = \frac{-2}{\pi}\frac{\cos 3x}{3}\Big|_0^{\pi}$$

$$= \frac{-2}{3\pi}(\cos 3\pi-\cos 0)$$

$$= \frac{-2}{3\pi}(-1-1) = \frac{4}{3\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\, cosnx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} sin(3x)\, cosnx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2sin(3x)\, cosnx\, dx$$

$$since\ 2sinAcosB = sin(A+B) + sin(A-B)$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{\pi}\big(sin(3x+nx) + sin(3x-nx)\big)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(sin(n+3)x - sin(n-3)x)\, dx$$

$$for\ n = 3, we\ have$$

$$a_3 = \frac{1}{\pi}\int_0^{\pi}(sin6x - sin0)\, dx = \frac{1}{\pi}\int_0^{\pi} sin6x\, dx$$

$$\therefore\ a_3 = \frac{-1}{\pi}\ \frac{cos6x}{6}\ \Big|_0^{\pi}$$

$$= \frac{-1}{6\pi}\ [cos6x - cos0] = \frac{-1}{6\pi}\ (1-1) = 0$$

$$\therefore a_3 = 0$$

$$for\ n = 2, 4, 5, \dots \dots$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}(sin(n+3)x - sin(n-3)x)\, dx$$

$$= \frac{1}{\pi}\left[\frac{-cos(n+3)x}{n+3} + \frac{cos(n-3)x}{n-3}\right]\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-cos(n+3)\pi + cos0}{n+3} + \frac{cos(n-3)\pi - cos0}{n-3}\right]$$

$$= \frac{1}{\pi}\left[\frac{cos(n-3)\pi}{n-3} - \frac{cos(n+3)\pi}{n+3} + \frac{1}{n+3} - \frac{1}{n-3}\right]$$

$$since\ cos(n-3)\pi = cos(n+3)\pi = -cosn\pi$$

$$\therefore\ a_n = \frac{1}{\pi}\left[\frac{-cosn\pi}{n-3} + \frac{cosn\pi}{n+3} + \frac{n-3-n-3}{n^2-9}\right]$$

$$= \frac{1}{\pi}\left[\frac{(n-3-n-3)cosn\pi}{n^2-9} - \frac{6}{n^2-9}\right]$$

$$\therefore\ a_n = \frac{1}{\pi}\left[\frac{-6cosn\pi}{n^2-9} - \frac{6}{n^2-9}\right]$$

$$\therefore\ a_n = \frac{-6}{\pi}\left[\frac{1+cosn\pi}{n^2-9}\right]$$

$$= \frac{-6}{\pi}\left[\frac{1+(-1)^n}{n^2-9}\right]$$

$\therefore$ ***The fourier cosine series is***

$$\frac{a_0}{2} + a_1cos2x + a_2cos2x + a_3cos3x + \sum_{n=4}^{\infty} a_n\,cosnx$$

$$since\ a_0 = \frac{4}{3\pi}, \qquad a_1 = \frac{-6}{\pi}\left[\frac{1-1}{1-9}\right] = 0,$$

$$a_2 = \frac{-6}{\pi}\left[\frac{1+1}{4-9}\right] = \frac{-6}{\pi}\frac{2}{-5} = \frac{12}{5\pi}, \qquad a_3 = 0$$

$\therefore$ ***Hence, fourier cosine series is***

$$\frac{2}{3\pi} + \frac{12}{5\pi}cos2x - \frac{6}{\pi}\sum_{n=4}^{\infty}\frac{1+(-1)^n}{n^2-9}\,cosnx$$

or

$$\frac{2}{3\pi} + \frac{12}{5\pi}cos2x + \frac{6}{\pi}\sum_{n=4}^{\infty}\frac{1+(-1)^n}{9-n^2}\,cosnx$$