SECTION 1.3

Problem 1

$3yu_x - 2xu_y = 0$

$(\boldsymbol{a})\ \boldsymbol{u(x,y) = x^2 on\ the\ line\ y = x}$

$(\boldsymbol{b})\ \boldsymbol{u(x,y) = 1 - x^2\ on\ the\ line\ y = -x}$

$(\boldsymbol{c})\ \boldsymbol{u(x,y) = 2x\ on\ the\ ellipse\ 3y^2 + 2x^2 = 4}$

Solution

$3yu_x - 2xu_y = 0 \dots\dots\dots (1)$

Characteristics Equation

$$\frac{dy}{dx} = \frac{-2x}{3y}$$

$$3ydy + 2xdx = 0$$

$$\frac{3y^2}{2} + x^2 = c_1$$

$$3y^2 + 2x^2 = c_2 = \psi(x,y)$$

Transformation

$$u(x,y) = \boldsymbol{\omega(\xi,\eta)}$$

$\boldsymbol{\xi = x}$ $\qquad \boldsymbol{\eta = 3y^2 + 2x^2}$

$\boldsymbol{\xi_x = 1}$ $\qquad \boldsymbol{\xi_y = 0}$

$\boldsymbol{\eta_x = 4x}$ $\qquad \boldsymbol{\eta_y = 6y}$

$$\boldsymbol{u_x = \omega_\xi \xi_x + \omega_\eta \eta_x = \omega_\xi + 4x\omega_\eta}$$

$$\boldsymbol{u_y = \omega_\xi \xi_y + \omega_\eta \eta_y = 0 + 6y\omega_\eta}$$

$\boldsymbol{substitute\ for\ u_x\ and\ u_y\ in\ equation\ 1}$

$$3y(\omega_\xi + 4x\omega_\eta) - 2x(6y\omega_\eta) = 0$$

$$3y\omega_\xi + 12xy\omega_\eta - 12xy\omega_\eta = 0$$

$$3y\omega_\xi = 0$$

$$\Rightarrow \omega_\xi = 0$$

Integrate, we have

$$\omega = f(\eta)$$

<u>*Transformation*</u>

$$u(x,y) = \omega(\xi,\eta) = f(\eta)$$

$$u(x,y) = f(3y^2 + 2x^2)$$

$\therefore$ ***The general solution is***

$$u(x,y) = f(3y^2 + 2x^2)$$

where f is a differentiable function of one variable.

(a) ***Now*** $u(x,y) = x^2$ ***on the line*** $y = x$

$$\Rightarrow u(x,x) = x^2 = f(3x^2 + 2x^2) = f(5x^2)$$

$$\therefore f(5x^2) = x^2$$

let $t = 5x^2$

$$\Rightarrow x^2 = \frac{t}{5}$$

$$\therefore f(t) = \frac{t}{5}$$

$$\therefore u(x,y) = \frac{3y^2+2x^2}{5}$$

(b) $u(x,y) = 1 - x^2$ ***on the line*** $y = -x$

$$u(x,-x) = 1 - x^2 = f\left(3x^2 + 2x^2\right) = f\left(5x^2\right)$$

$$\therefore f\left(5x^2\right) = 1 - x^2.$$

$$let\ t = 5x^2$$

$$\Rightarrow x^2 = \frac{t}{5}$$

$$i.e\ f(t) = 1 - \frac{t}{5}$$

$$\therefore u(x,y) = 1 - \frac{3y^2+2x^2}{5}$$

$$= \frac{5-3y^2+2x^2}{5}$$

(c) $u(x,y) = 2x$ *on the ellipse* $3y^2 + 2x^2 = 4$

$$u(x,y) = u\left(x, \pm\sqrt{\frac{4-2^{x2}}{3}}\right) = 2x$$

$$f = (4)$$

$$\Rightarrow f(4) = 2x.$$

This is impossible because $2x$ ***is not constant.***

Hence there is no solution on the ellipse

$3y^2 + 2x^2 = 4$.

<u>Problem 2</u>

$$u_x - 6u_y = y$$

(a) $u(x,y) = e^x$ *on the* $y = -6x + 2$

(b) $u(x,y) = 1$ *on the parabola* $y = -x^2$

(c) $u(x,y) = -4x$ *on the line* $y = -6x$

<u>Solution</u>

$u_x - 6u_y = y \dots\dots(2)$

<u>*Characteristic Equation*</u>

$\frac{dy}{dx} = \frac{-6}{1}$

$dy = -6dx$

Integrate, we have

$y + 6x = c = \boldsymbol{\psi(x, y)}$

<u>***Transformation***</u>

$\boldsymbol{u(x, y) = \omega(\xi, \eta)}$

$\boldsymbol{\xi = x, \qquad \eta = y + 6x}$

$\boldsymbol{\xi_x = 1 \qquad \xi_y = 0}$

$\boldsymbol{\eta_x = 6x \qquad \eta_y = 1}$

$\boldsymbol{u_x = \omega_\xi \xi_x + \omega_\eta \eta_x = \omega_\xi + 6\omega_\eta}$

$\boldsymbol{u_y = \omega_\xi \xi_y + \omega_\eta \eta_y = 0 + \omega_\eta}$

substitute into $\boldsymbol{(1)}$

$\boldsymbol{\omega_\xi + 6\omega_\eta - 6\omega_\eta = y}$

$\boldsymbol{\omega_\xi = y = \eta - 6\xi}$

$\boldsymbol{\omega_\xi = \eta - 6\xi}$

Integrate, we have

$\boldsymbol{\omega = \eta\xi - 3\xi^2 + g(\eta)}$

$\boldsymbol{u(x, y) = \omega(\xi, \eta) = (y + 6x)x - 3x^2 + g(y + 6x)}$

$u(x,y) = xy + 6x^2 - 3x^2 + g(y+6x)$

$= xy + 3x^2 + g(y+6x)$

$= x(y+3x) + g(y+6x)$

$\therefore$ *The general solution is*

$u(x,y) = x(y+3x) + g(y+6x)$

where g is a differentiable function of one variable.

$(a)\ u(x,y) = e^x$ *on the* $y = -6x + 2$

$u(x, -6x+2) = e^x = x(-6x+2+3x) + g(-6x+2+6x)$

$\Rightarrow e^x = x(-3x+2) + g(2)$

$\therefore g(2) = e^x - x(-3x+2)$

This is impossible because $e^x - x(2-3x)$

is not constant. Hence there is no solution

on the line $y = -6x + 2$.

$(b) u(x,y) = 1$ *on the parabola* $y = -x^2$

$u(x, -x^2) = 1 = x(-x^2+3x) + g(-x^2+6x)$

$\Rightarrow 1 = x^2(3-x) + g(6x - x^2)$

$\therefore g(6x - x^2) = 1 + x^3 - 3x^2$

let $t = 6x - x^2$

$\Rightarrow x^2 - 6x + t$

$x = \frac{6 \pm \sqrt{36-4t}}{2} = 3 \pm \sqrt{9-t}$

$\therefore x = 3 + \sqrt{9-t}$ *or* $3 - \sqrt{9-t}$

$$\therefore\ g(t) = 1 + \left(3 + \sqrt{9-t}\ \right)^3 - 3\left(3 + \sqrt{9-t}\ \right)^2$$

or

$$g(t) = 1 + \left(3 - \sqrt{9-t}\ \right)^3 - 3\left(3 - \sqrt{9-t}\ \right)^2$$

$$\therefore\ u(x,y) = x(y+3x) + 1 + \left(3 + \sqrt{9-y-6x}\ \right)^3$$

$$-3\left(3 + \sqrt{9-y-6x}\ \right)^2$$

$$\therefore\ u(x,y) = 1 + x(y+3x) + \left(3 + \sqrt{9-y-6x}\ \right)^2$$

$$+\left(3 + \sqrt{9-y-6x} - 3\right)$$

$$\therefore\ u(x,y) = 1 + x(y+3x) + \left(3 + \sqrt{9-y-6x}\ \right)^2 \left(\sqrt{9-y-6x}\ \right)$$

Similarly,

$$for\ \ g(t) = 1 + \left(3 - \sqrt{9-t}\ \right)^3 - 3\left(3 - \sqrt{9-t}\ \right)^2$$

$$u(x,y) = x(y+3x) + 1 + \left(3 - \sqrt{9-y-6x}\ \right)^3$$

$$-3\left(3 - \sqrt{9-y-6x}\ \right)^2$$

$$u(x,y) =$$

$$x(y+3x) + 1 + \left(3 - \sqrt{9-y-6x}\ \right)^2\left(3 - \sqrt{9-y-6x} - 3\right)$$

$$\Rightarrow\quad u(x,y) = 1 + x(y+3x) + \left(3 - \sqrt{9-y-6x}\ \right)^2\left(-\sqrt{9-y-6x}\ \right)$$

$$(c)\ u(x,y) = -4x\ on\ the\ line\ y = -6x$$

$$u(x,-6x) = -4x = x(-6x+3x) + g(-6x+6x)$$

$$\Rightarrow -4x = x(-3x) + g(0)$$

$$-4x = -3x^2 + g(0)$$

$\therefore g(0) = 3x^2 - 4x$

This is impossible because $3x^2 - 4x$

is not constant. Hence there is no solution

on the line $y = -6x$.

Problem 3

$4u_x + 8u_y - u = 1$

(a) $u(x, y) = \cos(x)$ *on the line* $y = 3x$

(b) $u(x, y) = x$ *on the line* $y = 2x$

(c) $u(x, y) = 1 - x$ *on the curve* $y = x^2$

Solution

$4u_x + 8u_y - u = 1 \ldots\ldots\ldots (3)$

Characteristic equation

$$\frac{dy}{dx} = \frac{8}{4} = 2$$

$$dy = 2dx$$

integrate

$$y - 2x = c = \psi(x, y)$$

Transformation

$$u(x, y) = \omega(\xi, \eta)$$

$$\xi = x, \qquad \eta = y - 2x$$

$$\xi_x = 1 \qquad \xi_y = 0$$

$$\eta_x = -2 \qquad \eta_y = 1$$

$$u_x = \omega_\xi \xi_x + \omega_\eta \eta_x = \omega_\xi - 2\omega_\eta$$

$$u_y = \omega_\xi \xi_y + \omega_\eta \eta_y = 0 + \omega_\eta$$

substitute into (3)

$$4(\omega_\xi - 2\omega_\eta) + 8(\omega_\eta) - \omega = 1$$

$$4\omega_\xi - 8\omega_\eta + 8\omega_\eta - \omega = 1$$

$$4\omega_\xi - \omega = 1$$

$$\omega_\xi - \frac{\omega}{4} = \frac{1}{4} \dots\dots\dots (4)$$

<u>Integrating factor</u> $\underline{K(\xi)}$

$$K(\xi) = e^{\int -1/4 d\xi} = e^{-\xi/4}$$

Multiply (4)***by*** $K(\xi)$

$$e^{-\xi/4}\omega_\xi - e^{-\xi/4}\frac{\omega}{4} = \frac{e^{-\xi/4}}{4}$$

$$\frac{d}{d\xi}\left(\omega e^{-\xi/4}\right) = e^{-\xi/4}\frac{\omega}{4}$$

Integrate

$$\omega e^{-\xi/4} = -4e\frac{-\xi/4}{4} + g(\eta)$$

$$= -e^{-\xi/4} + g(\eta)$$

$$\therefore \omega = -1 + e^{\xi/4} g(\eta)$$

$$u(x,y) = \omega(\xi,\eta) = -1 + e^{\xi/4} + g(\eta)$$

$$u(x,y) = -1 + e^{x/4} + g(y - 2x)$$

The general solution is $u(x,y) = -1 + e^{x/4} g(y - 2x)$

where g is a differentiable function of one variable

$(a)\ u(x,y) = cos(x)\ on\ the\ line\ y = 3x$

$u(x,3x) = -1 + e^{x/4}g(3x - 2x) = cosx$

$-1 + e^{x/4}g(x) = cosx$

$\therefore\ g(x) = (1 + cosx)e^{-x/4}$

$let\ x = t$

$g(t) = (1 + cost)e^{-t/4}$

$\therefore\ u(x,y) = -1 + e^{x/4}\big(1 + cos(y - 2x)\big)e^{\frac{-y+2x}{4}}$

$= -1 + e^{x/4}e^{-y/4}e^{2x/4}\big(1 + cos(y - 2x)\big)$

$= -1 + e^{3x/4}e^{-y/4}\big(1 + cos(y - 2x)\big)$

$= -1 + e^{(3x-y)/4}\big(1 + cos(y - 2x)\big)$

$(b)\ u(x,y) = x\ on\ the\ line\ y = 2x$

$u(x,2x) = x = -1 + e^{x/4}g(2x - 2x)$

$\therefore\ x = -1 + e^{x/4}g(0)$

$g(0) = (1 + x)e^{-x/4}$

$This\ is\ impossible\ because\ (1 + x)e^{-x/4}$

$is\ not\ constant. Hence\ there\ is\ no\ solution$

$on\ the\ line\ y = 2x.$

$(c)\ u(x,y) = 1 - x\ on\ the\ curve\ y = x^2$

$u(x,x^2) = 1 - x = -1 + e^{x/4}g(x^2 - 2x)$

$\therefore\ 1 - x = -1 + e^{x/4}g(x^2 - 2x)$

$\therefore\ g(x^2 - 2x) = (2 - x)e^{-x/4}$

$$let\ t = x^2 - 2x$$

$$x^2 - 2x - t = 0$$

$$\therefore x = \frac{2 \pm \sqrt{4+4t}}{2}$$

$$= 1 \pm \sqrt{1+t}$$

$$\therefore x = 1 + \sqrt{1+t} \quad or \quad 1 - \sqrt{1+t}$$

$$\therefore g(t) = \left(2 - 1 - \sqrt{1+t}\right)e^{-(1+\sqrt{1+t}\,)/4}$$

$$= \left(1 - \sqrt{1+t}\right)e^{-(1+\sqrt{1+t}\,)/4}$$

$$or$$

$$g(t) = \left(2 - 1 + \sqrt{1+t}\right)e^{-\left(1-\sqrt{1+t}\right)/4}$$

$$= \left(1 + \sqrt{1+t}\right)e^{e^{-\left(1-\sqrt{1+t}\right)/4}}$$

$$\therefore u(x,y) = -1 + e^{x/4}\left(1 - \sqrt{1+y-2x}\right)e^{-\left(1+\sqrt{1+y-2x}\right)/4}$$

$$or$$

$$u(x,y) = -1 + e^{x/4}\left(1 + \sqrt{1+y-2x}\right)e^{-\left(1-\sqrt{1+y-2x}\right)/4}.$$