Section 1.1

Problem 2

$Let\ c\ be\ a\ positive\ constant. Show\ that$

$u(x,t) = f(x+ct) + g(x-ct)\ is\ a\ solution\ of$

$u_{tt} = c^2 u_{xx}$

$for\ any\ twice-differentiable\ functions\ f\ and\ g\ of\ one\ variable.$

Solution

$u(x,t) = f(x+ct) + g(x-ct)$

$u_t = cf'(x+ct) - cg'(x+ct)$

$u_{tt} = c^2 f''(x+ct) + c^2 g''(x+ct) \ldots\ldots\ldots..1$

$u_x = f'(x+ct) + cg'(x+ct)$

$u_{xx} = f''(x+ct) + g''(x+ct) \ldots\ldots.\ldots..2$

$Multiply\ equation\ 2\ by\ c^2\ we\ have:$

$c^2 u_{xx} = c^2 f''(x+ct) + c^2 g''(x+ct) \ldots\ldots.\ldots..3$

$Compare\ equation\ 1\ with\ equation\ 3\ we\ have:$

$u_{tt} = c^2 u_{xx}$

$proved$

Problem 4

$Show\ that\ if\ p\ is\ a\ continuously\ differentiable\ function\ of\ one\ variable,$

$the\ first-order\ partial\ differential\ equation\ \ u_t = p(u)u_x$

$has\ a\ solution\ implicitly\ defined\ by\ u(x,t) = \varphi(x + p(u)t),$

$in\ which\ \varphi\ \ is\ any\ continously\ differentiable\ function\ of\ one\ variuable.$

$Use\ this\ idea\ to\ determine(perhaps\ implicitly)a\ solution\ of\ each\ of\ the$

$following\ equations.$

$(a) u_t = k u_x$

(b)$u_t =$ uu_x

(c)$u_t = cos(u) u_x$

(d)$u_t = e^u u_x$

(e)$u_t =$ u$sin(u) u_x$

<u>*Solution*</u>

$u(x,t) = \varphi(x + p(u)t)$

$u_t = \varphi'(x + p(u)t)\{p'(u)u_t t + p(u)\} \ldots \ldots .4$

$u_x = \varphi'(x + p(u)t)\{1 + p'(u)u_x t\} \ldots \ldots .5$

$Multiply\ equation\ 5\ by\ p(u) we\ have:$

$p(u)u_x = \varphi'(x + p(u)t)\{p(u) + p(u)p'(u)u_x t\} \ldots \ldots .6$

$u_t - p(u)u_x = \varphi'(x + p(u)t)\{p'(u)u_t t + p(u)\} - \varphi'(x + p(u)t)$

$\{p(u) + p(u)p'(u)u_x t\}$

$u_t - p(u)u_x = \varphi'(x + p(u)t)[\{p'(u)u_t t + p(u)\} - \{p(u) + p(u)p'(u)u_x t\}]$

$u_t - p(u)u_x =\varphi'(x + p(u)t)[p'(u)u_t t + p(u) - p(u) - p(u)p'(u)u_x t\}]$

$u_t - p(u)u_x =\varphi'(x + p(u)t)[p'(u)u_t t - p(u)p'(u)u_x t\}]$

$u_t - p(u)u_x =\varphi'(x + p(u)t)[u_t - p(u)u_x t\}]p'(u)t$

$u_t - p(u)u_x - \varphi'(x + p(u)t)[u_t - p(u)u_x t\}]p'(u)t=0$

$\{u_t - p(u)u_x\}\{1 - \varphi'(x + p(u)t)p'(u)t\}=0$

Since $\{1 - \varphi'(x + p(u)t)p'(u)t\} \neq 0$

$\Rightarrow u_t - p(u)u_x=0$

Hence $u_t = p(u)u_x$

$(a) u_t = ku_x$ here p(u) = k

$Hence\ u(x,t) = \varphi(x + kt)$

(b)$u_t = \mathrm{u}u_x$, here p(u) = u

$Hence\ u(x,t) = \varphi(x + ut)$

(c)$u_t = cos(u)u_x$, here p(u) = cos(u)

$Hence\ u(x,t) = \varphi(x + \cos(u)t)$

(d)$u_t = e^u u_x$, here p(u) = e^{u}

$Hence\ u(x,t) = \varphi(x + \mathrm{e}^{\mathrm{u}} t)$

(e)$u_t = \mathrm{u}sin(u)u_x$, here p(u) = sin(u)

$Hence\ u(x,t) = \varphi(x + \mathrm{u}\sin(u)t)$

<u>Problem 7</u>

In each of the following, classify the equation as linear, quasi-linear and not linear, or not quasi linear.

(a) $u^2 u_{xx} + uy = \cos(u)$

$(b)\ x^2 u_x + y^2 u_y + u_{xy} = 2xy$

$(c)(x - y){u_x}^2 + u_{xy} = 1$

$(d)\ \ (x - y){u_x}^2 + \ 2u_y = 4y$

$(e)x^2 u_{yy} - yu_{xx} = \tan(u)$

(f) $u_x + \ {u_y}^2 - u_{xx} = 4$

(g) $u_x - u_x u_y - u_y = 0$

$(h)\ uu_x + u_{xy} = u^2$

$(i)u_{xy} - {u_x}^2 + {u_y}^2 - \sin(\mathrm{u_x}) = 0$

$(j)\quad \frac{u_y}{u_x} = x^2$

SOLUTION

(a) $u^2 u_{xx} + uy = \cos(u)$

It is not linear because of cos(u)

It is quasi-linear because it is linear in the highest order derivative u_{xx}

$(b)\ x^2 u_x + y^2 u_y + u_{xy} = 2xy$

$It\ is\ linear.$

$(c)(x-y){u_x}^2 + u_{xy} = 1$

$It\ is\ not\ linear\ because\ of\ \ {u_x}^2. It\ is\ quasi-linear\ because\ it\ is$

$linear\ in\ the\ highest\ order\ derivative\ u_{xy}.$

$(d)\ \ (x-y){u_x}^2 +\ 2u_y = 4y$

$It\ is\ not\ linear\ because\ of\ \ {u_x}^2. It\ is\ also\ not\ quasi-\ linear$

$because\ it\ is\ not\ linear\ in\ the\ highest\ order\ derivative\ {u_x}^2$

$(e)x^2 u_{yy} - yu_{xx} = \tan(u)$

$It\ is\ not\ linear\ becauseof\ \tan(u). It\ is\ quasi-linear\ because\ it\ is$
$linear\ in\ the\ highest\ order\ derivatives\ u_{yy}\ and\ u_{xx}.$

(f) $u_x +\ {u_y}^2 - u_{xx} = 4$

$It\ is\ not\ linear\ because\ of\ \ {u_y}^2.$

$It\ is\ quasi-linear\ because\ it\ is\ linear$
$in\ the\ highest\ order\ derivative\ u_{xx}.$

(g) $u_x - u_x u_y - u_y = 0$

$It\ is\ not\ linear\ because\ of\ u_x u_y.$

$It\ is\ also\ not\ quasi-linear\ because\ it\ is\ not\ linear\ in\ the$
$highest\ order\ derivative\ u_x u_y.$

$(h)\ uu_x + u_{xy} = u^2$

It is not linear because of u^2.

It is quasi − linear because it is linear in the highest order derivative u_{xy}.

$(i) u_{xy} - {u_x}^2 + {u_y}^2 - \sin(u_x) = 0$

It is not linear because of $\sin(u_x), {u_x}^2, {u_y}^2$

It is quasi − linear because it is linear in the highest order derivative u_{xy}.

$(j) \quad \frac{u_y}{u_x} = x^2$

It is not linear because of $\frac{u_y}{u_x}$. *It is also not quasi linear.*

Problem 8

Let k be a positive constant. Let

$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi$ *in which f is continuos on the real line.*

Show that $u_t = k u_{xx}$ *for* $-\infty < x < \infty, t > 0$.

Also determine $u(x,t)$ *when* $f(x) = 1$ *for all real x.*

Solution

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi$$

Let $2\sqrt{\pi k} = \omega$

Hence the equation becomes $u(x,t) = \frac{1}{\omega\sqrt{t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi$

$$u_t = -\frac{1}{2\omega\sqrt{t^3}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi + \frac{1}{\omega\sqrt{t}} \int_{-\infty}^{\infty} \frac{(x-\xi)^2}{4kt^2} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi$$

$$u_t = -\frac{1}{2\omega\sqrt{t^3}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi + \frac{1}{\omega\sqrt{t}} \frac{1}{4kt^2} \int_{-\infty}^{\infty} (x-\xi)^2 e^{-\frac{(x-\xi)^2}{4kt}} f(\xi) d\xi$$

$$u_t = -\frac{1}{2\omega\sqrt{t^3}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi + \frac{1}{4k\omega\sqrt{t^5}}\int_{-\infty}^{\infty}(x-\xi)^2 e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi \ldots..7$$

$$u_x = \frac{1}{\omega\sqrt{t}}\int_{-\infty}^{\infty}\left(-\frac{x-\xi}{2kt}\right)e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi$$

$$u_x = \frac{-1}{2k\omega\sqrt{t^3}}\int_{-\infty}^{\infty}(x-\xi)e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi$$

$$u_{xx} = \frac{-1}{2k\omega\sqrt{t^3}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi + \frac{1}{4k^2\omega\sqrt{t^5}}\int_{-\infty}^{\infty}(x-\xi)^2 e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi \ldots 8$$

$Multiply\ equation\ 8\ by\ k\ we\ have$:

$$ku_{xx} = \frac{-1}{2\omega\sqrt{t^3}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi + \frac{1}{4k\omega\sqrt{t^5}}\int_{-\infty}^{\infty}(x-\xi)^2 e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi \ldots 9$$

$Compare\ equation\ 7\ with\ equation\ 9\ we\ have$:

$$u_t = ku_{xx}$$

$When\ f(x) = 1\ for\ all\ real\ x, we\ have$:

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi$$

$Let\ \frac{x-\xi}{2\sqrt{kt}} = w$

$Then\ \frac{dw}{d\xi} = \frac{-1}{2\sqrt{kt}}$

$Hence\ d\xi = -2\sqrt{kt}\, dw$

$also\ the\ limit\ of\ the\ integral\ becomes$:

$\infty\ changes\ to\ -\infty$

$-\infty\ changes\ to\ \infty$

$$\therefore\ u(x,t) = \frac{-2\sqrt{kt}}{2\sqrt{\pi kt}}\int_{+\infty}^{-\infty} e^{-w^2}\, dw$$

$$\therefore\ u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-w^2}\, dw$$

Using the standard result that

$$\int_{-\infty}^{\infty} e^{-w^2}\, dw = \sqrt{\pi}\ ,$$

we have $u(x,t) = \frac{1}{\sqrt{\pi}}\sqrt{\pi}$

$$u(x,t) = \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$\therefore u(x,t) = 1$$