King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 470 Exam II
Semester I, 2009- (091)
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ID:	KEY
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Q		Points
1		20
2		15
3		20
4		13
5		20
6		12
Total		

☺ Say Bismillah & Good luck ☺

(1) State and prove the Maximum Principle Theorem for Harmonic Functions.

See textbook page 258 - 261

(2) Use separation of variable to solve the following equation

$$u_{tt} + 2u_t + u = c^2 u_{xx} \quad \text{for} \quad 0 < x < \pi, t > 0 \qquad (1)$$

in which c is positive constant. The boundary conditions are

$$u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t > 0 \qquad (2)$$

and the initial conditions are

$$u(x,0) = 1 \ \text{and} \ u_t(x,0) = -1 \quad \text{for} \quad 0 < x < \pi$$

[redacted]

attempt a solution of the form

$$u(x,t) = X(x) \cdot T(t)$$

Subsitute into equation, we get

$$\frac{T''}{c^2 T} + 2\frac{T'}{c^2 T} + \frac{1}{c^2} = \frac{X''}{X}$$

Proceeding as before (wave eq), we obtain

$$\boxed{\begin{matrix} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{matrix}} \quad \text{and} \quad \boxed{T'' + 2T' + (1+\lambda c^2)T = 0}$$

$\lambda < 0 \Rightarrow$ trivial solution.

If $\lambda = 0 \Rightarrow$ trivial solution,

If $\lambda > 0$, $\lambda = n^2 \Rightarrow X_n(x) = \sin(nx)$ with $\lambda = n^2$

Now the differential equation for T

$$T'' + 2T' + (1 + n^2 c^2)T = 0 \qquad (*)$$

The charac. equation is $m^2 + 2m + (1 + n^2c^2) = 0$

$$\Delta = 4 - 4(1+n^2c^2) = -4n^2c^2 \text{ is negative}$$

roots of charac. equation: $-1 \pm nci$

The general solution for $(*)$

$$T_n(t) = e^{-t}\left[a_n \cos(nct) + b_n \sin(nct)\right]$$

we therefore have

$$u_n(x,t) = X_n(x) \cdot T_n(t) = e^{-t}\left[a_n \cos(nct) + b_n \sin(nct)\right] \sin(nx)$$

which satisfy eq(1) and eq(2) for any choices of coefficients

To satisfy the initial condition,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} e^{-t}\left[a_n \cos(nct) + b_n \sin(nct)\right] \sin(nx)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} e^{-t}\left[-nca_n \sin(nct) + ncb_n \cos(nct)\right]\sin(nx)$$

$$-e^{-t}\left[a_n \cos(nct) + b_n \sin(nct)\right] \sin(nx)$$

$$u(x,0) = 1 \Rightarrow \sum_{n=1}^{\infty} a_n \sin(nx) = 1$$

This is the Fourier Sine expansion of $f(x)=1$ on $[0,\pi]$

So choose $a_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = -\frac{2}{n\pi}\left[\cos(nx)\right]_0^{\pi}$

$$= \frac{2}{\pi}\left[1-(-1)^n\right]$$

$$u_t(x,0) = -1 \Rightarrow \sum_{n=1}^{\infty} (ncb_n - a_n)\sin(nx) = -1$$

This is the Fourier Sine expansion of $f(x) = -1$

So, $(ncb_n - a_n) = \frac{2}{\pi}\int_0^{\pi} -\sin(nx)\,dx$

$$= \frac{2}{n\pi}\left[\cos(nx)\right]_0^{\pi} = \frac{-2}{n\pi}\left[1-(-1)^n\right]$$

$$\Rightarrow ncb_n - a_n = -a_n \Rightarrow ncb_n = 0 \Rightarrow b_n = 0$$

The solution is

$$\boxed{\begin{gathered} u(x,t) = e^{-t}\sum_{n=1}^{\infty} a_n \cos(nct)\sin(nx) \\ a_n = \frac{2}{\pi}\left[1-(-1)^n\right] \end{gathered}}$$

$$a_n = \begin{cases} 0 & n = \text{even} \\ 4/\pi & n = \text{odd} \quad n = 2k-1,\ k = 1, 2, \ldots \end{cases}$$

$$u(x,t) = e^{-t} \sum_{k=1}^{\infty} \frac{4}{\pi} \cos((2k-1)ct) \sin((2k-1)x)$$

$$\boxed{u(x,t) = \frac{4e^{-t}}{\pi} \sum_{k=1}^{\infty} \cos((2k-1)ct) \sin((2k-1)x)}$$

(3) Consider the following problems

Problem 1:
$u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty, t > 0$
$u(x,0) = \sin x$, $u_t(x,0) = \psi(x)$

Problem 2:
$u_{tt} = c^2 u_{xx}$ for $-\infty < x < \infty, t > 0$
$u(x,0) = \sin x + 0.01$, $u_t(x,0) = \psi(x)$

Let $v(t,x)$ be a solution of problem 1 and
$w(t,x)$ be a solution of problem 2, then
find an upper bound for the difference

$|v(t,x) - w(t,x)| \leq$ ....

(show all your work).

By d'Alembert formula

$$V(t,x) = \frac{1}{2}\left[\sin(x-ct) + \sin(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$w(t,x) = \frac{1}{2}\left[\sin(x-ct) + 0.01 + \sin(x+ct) + 0.01\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$V(t,x) - w(t,x) = -\frac{1}{2}\left[0.01 + 0.01\right] = -0.01$$

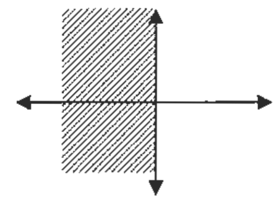
Hence, $|v(t,x) - w(t,x)| = 0.01$

or

$|v(t,x) - w(t,x)| \leq 0.01$

(4) Find the solution of the Drichlet problem for the left half-plane.

$$\Delta u = 0 \qquad \text{for} \quad -\infty < y < \infty, x < 0$$
$$u(0, y) = f(y) \qquad \text{for} \quad -\infty < y < \infty$$



where

$$f(y) = \begin{cases} 0 & \text{for } \; y < 0 \; \text{ and } \; y > 1 \\ y & \text{for } \quad 0 \le y \le 1 \end{cases}$$

By separation of variables, $u(x,y) = X(x) \cdot Y(y)$

we obtain $\dfrac{Y''}{Y} = -\dfrac{X''}{X} = -\lambda$

or $Y'' + \lambda Y = 0$, $X'' - \lambda X = 0$

Take cases on λ; we get if $\lambda > 0$, $\lambda = w^2$ then

$Y_w(y) = a_w \cos(wy) + b_w \sin(wy)$

is bounded solution

Now, consider the x-dependence, with $\lambda = w^2$ we have

$X'' - w^2 X = 0$

The bounded solution: $X_w(x) = e^{wx}$ $(x<0)$

For each $w \geq 0$, we have

$$u_w(x,y) = \left[a_w \cos(wy) + b_w \sin(wy)\right] e^{wx}$$

To satisfy the boundary condition, let

$$u(x,y) = \int_0^\infty u_w(x,y)\, dw.$$

Now, $u(0,y) = f(y) \Rightarrow f(y) = \int_0^\infty \left(a_w \cos(wy) + b_w \sin(wy)\right) dw$

This is the Fourier integral expansion of f; hence choose

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi \quad \text{and} \quad b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi$$

The solution can be written as

$$u(x,y) = \int_0^\infty \left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi\right)\cos(\omega y) + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi\right)\sin(\omega y)\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} \left(f(\xi)\cos(\omega\xi)\cos(\omega y) + f(\xi)\sin(\omega\xi)\sin(\omega y)\right)d\xi\, d\omega$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int_0^\infty \cos\omega(\xi-y)\, e^{\omega x}\, d\omega\right] f(\xi)\,d\xi$$

Apply integration by parts two times to get

$$\int \cos\omega(\xi-y)\, e^{\omega x}\, d\omega = -\frac{x}{x^2+(\xi-y)^2}$$

Hence, $$\boxed{u(x,y) = -\frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{x^2+(\xi-y)^2}\,d\xi} \quad \text{---} (*)$$

$(*)$ can be obtained by replacing each $\begin{Bmatrix} x \to y \\ y \to -x \end{Bmatrix}$ in upper half plane formula

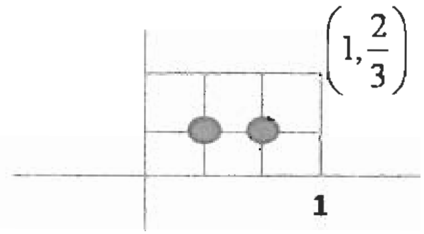
Now, $$u(x,y) = -\frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{x^2+(\xi-y)^2}\,d\xi = -\frac{x}{\pi}\int_0^1 \frac{\xi}{x^2+(\xi-y)^2}\,d\xi$$

Note: $$\int \frac{\xi}{x^2+(\xi-y)^2}\,d\xi = \left[\frac{1}{2}\ln\left(x^2+(\xi-y)^2\right) + \frac{y}{x}\tan^{-1}\left(\frac{\xi-y}{x}\right)\right]$$

$$u(x,y) = -\frac{x}{\pi}\left[\frac{1}{2}\ln\left(x^2+(\xi-y)^2\right) + \frac{y}{x}\tan^{-1}\left(\frac{\xi-y}{x}\right)\right]_{\xi=0}^{\xi=1}$$

(5) **use finite difference method to approximate the values $u\left(\frac{1}{3},\frac{1}{3}\right)$ and $u\left(\frac{2}{3},\frac{1}{3}\right)$ of the solution of the Laplace equation**

$$\Delta u = f(x,y) \quad \text{for} \quad (x,y) \in \Omega \qquad \text{—— (1)}$$
$$u = 0 \qquad\qquad \text{for} \quad (x,y) \in \partial\Omega$$



where

$$\Omega = \left\{(x,y): 0 < x < 1,\ 0 < y < \frac{2}{3}\right\}$$

$$f(x,y) = -2(x^2 + y^2)$$

(a) use the meshsize $h = \frac{1}{3}$. Then write the linear system AU=b.

(b) solve the linear system and write the approximate value of

$u\left(\frac{1}{3},\frac{1}{3}\right) \cong$ ....
$u\left(\frac{2}{3},\frac{1}{3}\right) \cong$ ....

Let $x_i = ih$, $y_j = jh$ $\quad i = 0,1,2,3$, $j = 0,1,2$

and $U_{ij} \cong u(x_i, y_j)$

The discritization of the PDE is

$$\frac{u(x_i+h, y_j) - 2u(x_i,y_j) + u(x_i-h, y_j)}{h^2} + \frac{u(x_i, y_j+h) - 2u(x_i,y_j) + u(x_i, y_j-h)}{h^2} \cong f(x_i, y_j)$$

$$\Rightarrow U_{i+1,j} + U_{i-1,j} - 4U_{ij} + U_{i,j+1} + U_{i,j-1} = h^2 f_{ij}$$

$i=1$; $j=1$: $U_{21} + U_{01} - 4U_{11} + U_{12} + U_{10} = h^2 f_{11}$

$i=2$, $j=1$: $U_{31} + U_{11} - 4U_{21} + U_{22} + U_{20} = h^2 f_{21}$ $\quad (*)$

$f_{11} = f(x_1, y_1) = -2\left(\frac{1}{9} + \frac{1}{9}\right) = -4/9$

$f_{21} = f(x_2, y_1) = -2\left(4/9 + 1/9\right) = -10/9$

Let $U = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$ then $(*)$ becomes:

$$\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} -4/81 \\ -10/81 \end{bmatrix}$$

Note.
$U_{00} = U_{10} = U_{20} = U_{30} = 0$
$U_{02} = U_{12} = U_{22} = U_{32} = 0$
$U_{01} = U_{31} = 0$

$$\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}^{-1} = -\frac{1}{15}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \left(-\frac{1}{15}\right)\left(-\frac{1}{81}\right)\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \frac{1}{1215}\begin{bmatrix} 26 \\ 44 \end{bmatrix} \Rightarrow \begin{array}{l} u\left(\frac{1}{3},\frac{1}{3}\right) \cong U_{11} = 26/1215 \\ u\left(\frac{2}{3},\frac{1}{3}\right) \cong U_{21} = 44/1215 \end{array}$$

(6) Let w be a solution of thre problem

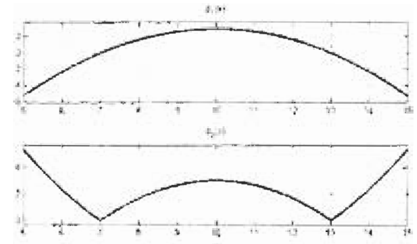
$$u_{tt} = u_{xx} \quad \text{for} \quad -\infty < x < \infty, t > 0$$

$$u(x,y) = \varphi_1(x), \quad u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty$$

Let u be a solution of

$$u_{tt} = u_{xx} \quad \text{for} \quad -\infty < x < \infty, t > 0$$

$$u(x,y) = \varphi_2(x), \quad u_t(x,0) = 0 \quad \text{for} \quad -\infty < x < \infty$$



where

$$\varphi_1(x) = -\frac{1}{3}x^2 + \frac{20}{3}x - \frac{91}{3}$$

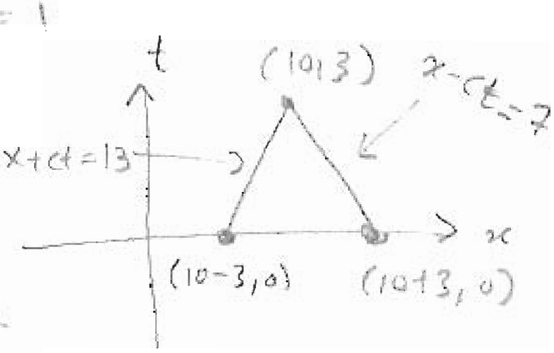
$$\varphi_2(x) = \left| -\frac{1}{3}x^2 + \frac{20}{3}x - \frac{91}{3} \right|$$

Find: $\boxed{|v(10,3) - w(10,3)| = \text{.......................}}$

(show all your work)

Note that $\varphi_1(x) = \varphi_2(x) \quad \forall x \in [7, 13]$

eq (1) is wave equation with $c = 1$



$[x_0 - ct_0, x_0 + ct_0] = [7, 13]$

is the domain of dependence of the point $(10,3)$ for both problems

Hence, $v(10,3) = w(10,3)$

$\Rightarrow |v(10,3) - w(10,3)| = 0$