King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 470 Final Exam
Semester I, 2009- (091)
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(1) Find the general solution of the given partial differential equation $xu_x - yu_y + u = x$

(2) Solve
$$u_x - 6u_y = y$$
$$u(x,y) = e^x \quad \text{on the line} \quad y = -6x + 2$$

(3) Solve $u_x - y^2 u_y = 1; \Gamma$ is given by $y = x^2 + 2; u = 0$

(4) Classify the equations as hyperbolic, parabolic, or elliptic (in a region of the plane where the coefficients are continuous).

(a) $4u_{xx} + u_{xy} - 2u_{yy} - \cos(xy) = 0$

(b) $yu_{xx} + 4u_{xy} + 4xyu_{yy} - 3u_y + u = 0$

(c) $u_{xy} - 2u_{xx} + (x+y)u_{yy} - xyu = 0$

(5) Find the canonical form

(a) $3u_{xx} + 6u_{xy} + 2u_{yy} = 0$

(b) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$

(c) $4u_{xx} + 2u_{xy} + u_{yy} - u_x + xyu_y + u = 0$

(6)Consider the wave equation
$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty$$
for the given $\varphi(x) = x^2, \psi(x) = \sin(2x), c = 4$.

(a) Solve the wave equation.
(b) Draw the characteristic triangle at the point (5,10).
(c) Find the domain of dependence of the point (5,10). Why we call this a domain of dependence?
(d) Draw the characteristic lines of this wave equation.

(7) Solve this equation
$$\nabla^2 u(x,y) = 0 \text{ for } -\infty < x < \infty, y > 0$$
$$u(x,0) = f(x) \text{ for } -\infty < x < \infty$$
where $f(x) = \begin{cases} 0 & \text{for } |x| > c \\ k & \text{for } -c \le x \le c \end{cases}$,
with k constant and c a positive constant.

(8) Use the conformal mapping method to write an integral solution for the Dirichlet problem for the upper-half plane. Compare this expression with equation 6.16.
$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\, d\xi$$

(9) Given the problem
$$u'' + u = f(x)$$
$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 0.$$
Find the Green's function

(10) (a) State the maximum principle theorem for the harmonic function.
(b) State the Mean value theorem for the harmonic function.

(11) Derive a variational formulation of the following problem:
$$-\Delta u + cu = f \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$
, **where** $c > 0, \quad \Omega \subset R^2$.

<u>BONUS QUESTION 1:</u> Solve the initial value boundary problem
$$u_t = ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0$$
$$u(0,t) = u(L,t) = 0 \text{ for } t \ge 0$$
$$u(x,0) = f(x) \text{ for } 0 \le x \le L$$
, **where** $F(x,t) = t, f(x) = x(L-x)$.

<u>BONUS QUESTION 2</u>: Use PDE tool box in Matlab to solve
$$\Delta u + 5u = f(x) \quad \Omega = (0,1)\times(0,\pi)$$
$$u(0,y) = u(1,y) = 0 \text{ for } 0 \le y \le \pi$$
$$\left.\begin{array}{l} u(x,0) = \sin(\pi x) \\ u(x,\pi) = 0 \end{array}\right\} \text{ for } 0 \le x \le 1$$
$$f(x) = 1$$
, **then approximate the value of** $u\left(\frac{1}{2}, \frac{\pi}{2}\right)$.