King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 470 Exam III
Semester I, 2009- (091)
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ID:	
Name:	

Q		Points
1		14
2		14
3		14
4		14
5		20
6		25
7		13
Bonus		25
Total		

☺ **Say Bismillah & Good luck** ☺

(1) Solve the Dirichlet problem **(#4/pg271)**

$$\nabla^2 u(x,y) = 0 \quad \text{for} \quad x^2 + y^2 < 9$$

$$u(x,y) = x^2 \quad \text{for} \quad x^2 + y^2 = 9$$

Hint: Convert the problem to polar coordinates.

(2) Find a bounded solution of the Dirichlet problem for the quarter-plane x<0, y<0. **(#2/pg282)**

$$\nabla^2 u = 0 \quad \text{for} \quad x < 0, y < 0$$
$$u(0, y) = 0 \quad \text{for} \quad y < 0$$
$$u(x,0) = f(x) \quad \text{for} \quad x < 0$$

(3) Consider the Newman problem

$$\left.\begin{array}{ll}\Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = f & \text{on } \partial\Omega\end{array}\right\} *$$

Drive necessary conditions for the existence of a solution of (*).

[Hint: Green's First identity $\int\limits_{\partial\Omega} g \dfrac{\partial h}{\partial n} ds = \iint\limits_{\Omega} (g\Delta h + \nabla g \cdot \nabla h) dA$].

(4) Solve **(#1/pg290)**

$$\nabla^2 u(x,y) = 0 \qquad \text{for} \quad 0 < x < 1, 0 < y < 1$$

$$u_x(0,y) = u_x(1,y) = 0 \qquad \text{for} \quad 0 \le y \le 1$$

$$u_y(x,0) = 4\cos(\pi x), u_y(x,1) = 0 \quad \text{for} \quad 0 \le x \le 1.$$

(5) Prove the Schwarz inequality for an inner product space: If x and y are vectors, then

$$|< x, y >| \leq \|x\| \|y\|$$ **(#1/pg343)**

Hint: Begin by expanding the right side of the inequality $0 \leq < x - \alpha y, x - \alpha y >$ and then choose $\alpha = < x, x > / < x, y >$, assuming that $< x, y > \neq 0$.

(6) Consider the following problem:

$$
\begin{aligned}
-u''+u &= f \quad \text{in} \quad \Omega=(0,1) \\
u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Give varitational formulation and investigate existence and uniqueness of solution of this problems.

(7) Derive a varitational formulation of the following problem:

$$
\begin{aligned}
-\Delta u + b \cdot \nabla u = f \quad & \text{in} \quad \Omega = (0,1) \times (0,1) \\
u = 0 \quad & \text{on} \quad \partial\Omega
\end{aligned}
$$

where

$$b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

BONUS QUESTION

$$\begin{aligned} -u''(x) &= f(x) \quad \text{in } \Omega=(0,1) \\ u(x) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Use Finite Element Method with h=1/3 and piecewise linear functions to compute the matrix A and the vector b then solve the linear system and write the solution as a linear combination of the basis then approximate the value at x= 0.3, 0.7. where f(x)= x(x-1).

$$\begin{bmatrix} a(\varphi_1,\varphi_1) & a(\varphi_2,\varphi_1) & \cdots & a(\varphi_n,\varphi_1) \\ a(\varphi_1,\varphi_2) & a(\varphi_2,\varphi_2) & \cdots & a(\varphi_n,\varphi_2) \\ \vdots & \vdots & \ddots & \vdots \\ a(\varphi_1,\varphi_n) & a(\varphi_2,\varphi_n) & \cdots & a(\varphi_n,\varphi_n) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} F(\varphi_1) \\ F(\varphi_2) \\ \vdots \\ F(\varphi_n) \end{bmatrix} \qquad Ax=b$$

NAME	FORMULA
The Fourier Series	$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$ $a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx$ $a_n=\frac{1}{L}\int_{-L}^{L}\cos(\frac{n\pi x}{L})f(x)dx \quad \text{for} \quad n=0,1,\ 2,\ ...$ $b_n=\frac{1}{L}\int_{-L}^{L}\sin(\frac{n\pi x}{L})f(x)dx \quad \text{for} \quad n=1,\ 2,\ ...$
The Fourier Sine Series	$f(x)=\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})$ $b_n=\frac{2}{L}\int_{0}^{L}\sin(\frac{n\pi x}{L})f(x)dx \quad \text{for} \quad n=1,\ 2,\ ...$
The Fourier Cosine Series	$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})$ $a_0=\frac{2}{L}\int_{0}^{L}f(\xi)d\xi$ $a_n=\frac{2}{L}\int_{0}^{L}\cos(\frac{n\pi x}{L})f(x)dx \quad \text{for} \quad n=0,1,\ 2,\ ...$
The Fourier Integral	$f(x)=\int_{0}^{\infty}\left[A_w\cos(wx)+B_w\sin(wx)\right]dw$ $A_w=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\cos(w\xi)d\xi$ $B_w=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\xi)\sin(w\xi)d\xi$
The Fourier Sine Integral	$f(x)=\int_{0}^{\infty}\left[B_w\sin(wx)\right]dw$ $B_w=\frac{2}{\pi}\int_{0}^{\infty}f(\xi)\sin(w\xi)d\xi$
The Fourier Cosine Integral	$f(x)=\int_{0}^{\infty}\left[A_w\cos(wx)\right]dw$ $A_w=\frac{2}{\pi}\int_{0}^{\infty}f(\xi)\cos(w\xi)d\xi$
Ends of the Bar Maintained at Temperature Zero	$u(x,t)=\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})\,e^{-\frac{n^2\pi^2kt}{L^2}}$ $b_n=\frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$
Temperature in a Bar with Insulated Ends	$u(x,t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})\,e^{-\frac{n^2\pi^2kt}{L^2}}$ $a_0=\frac{2}{L}\int_{0}^{L}f(\xi)d\xi$ $a_n=\frac{2}{L}\int_{0}^{L}f(\xi)\cos(\frac{n\pi\xi}{L})d\xi \quad \text{for} \quad n=0,1,\ 2,\ ...$

Ends of the Bar at Different Temperatures	$u(x,t) = U(x,t) + \frac{1}{L}(T_2 - T_1)x + T_1$ $U(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})\, e^{-\frac{n^2\pi^2 kt}{L^2}}$ $b_n = \frac{2}{L}\int_0^L f(\xi)\sin(\frac{n\pi\xi}{L})d\xi + 2\frac{(-1)^n T_2 - T_1}{n\pi}$ for $n = 0,1,\ 2,\ ...$
Diffusion of charges in a Transistor	$h(x,t) = e^{\alpha x+\beta t}u(x,t), \alpha = \frac{a}{2L}$ *and* $\beta = -\frac{ka^2}{4L^2}$ $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})\, e^{-\frac{n^2\pi^2 kt}{L^2}}$ $b_n = \frac{8KL}{ka}\frac{n\pi\,(1-e^{-a})}{4n^2\pi^2 a^2}$
The Heat Equation on the Real Line **(By Separation of Variable)**	$u(x,t) = \int_0^{\infty}[a_\omega \cos(\omega x) + b_\omega sin(\omega x)]e^{-\omega^2 kt}d\omega$ $a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\,\xi)d\xi$ $b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\,\xi)d\xi$
The Heat Equation on the Real Line **(By Fourier Transform)**	$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)d\xi$ $\int_{-\infty}^{\infty} e^{-\zeta^2}\, d\zeta = \sqrt{\pi}$
The Heat Equation on the Half Line **(By Separation of Variable)**	$u(x,t) = \int_0^{\infty} b_\omega sin(\omega x)\, e^{-\omega^2 kt}d\omega$ $b_\omega = \frac{2}{\pi}\int_0^{\infty} f(\xi)\sin(\omega\,\xi)d\xi$
The Heat Equation on the Half Line **(By Fourier Transform)**	$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_0^{\infty}[e^{-\frac{(x-\xi)^2}{4kt}} - e^{-\frac{(x+\xi)^2}{4kt}}]f(\xi)d\xi$
The Nonhomogeneous Heat Equation	$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(\frac{n\pi x}{L})$ $T_n(t)\int_0^t e^{-\frac{kn^2\pi^2(t-\tau)}{L^2}}B_n(\tau)d\tau + b_n e^{-\frac{kn^2\pi^2 t}{L^2}}$ $B_n(t) = \frac{2}{L}\int_0^L F(\xi,t)\sin(\frac{n\pi\xi}{L})d\xi$ $b_n = \frac{2}{L}\int_0^L f(\xi)\sin(\frac{n\pi\xi}{L})d\xi$
The Laplace Transform **The Fourier Transform** **Inverse Fourier Transform** **Operational Formula for Fourier Transform**	$L[f(x)](w) = \int_0^{\infty} e^{-wx} f(x)dx$ $L[f(x)](w) = \int_0^{\infty} e^{-wx} f(x)dx$ $F^{-1}[\hat{f}(w)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwx}\hat{f}(w)dw$ $F[f^{(n)}](w) = (iw)^n \hat{f}(w)$
Dirichlet Problem for the Upper Half-Plane	$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{y^2+(\xi-x)^2}d\xi$

Dirichlet Problem for the Right Quater	$u(x,y)=\frac{y}{\pi}\int_0^{\infty}\left[\frac{1}{y^2+(\xi-x)^2}-\frac{1}{y^2+(\xi+x)^2}\right]f(\xi)d\xi$
Dirichlet Problem for a Disk	$u(r,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n r^n\cos(n\theta)+b_n\ \ r^n\sin(n\theta)$ $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\xi)d\xi$ $a_n=\frac{1}{\rho^n L}\int_{-\pi}^{\pi}\cos(\frac{n\pi\xi}{L})f(\xi)d\xi \quad \text{for} \quad n=0,1,\ 2,\ ...$ $b_n=\frac{1}{\rho^n\pi}\int_{-\pi}^{\pi}\sin(\frac{n\pi\xi}{L})f(\xi)d\xi \quad \text{for} \quad n=1,\ 2,\ ...$
$Green's\ First\ Identity\ in\ R^2$	$\oint_{\partial\Omega}g\frac{\partial h}{\partial n}ds=\iint_{\Omega}(g\nabla^2 h+\nabla g\cdot\nabla h)dA$
Neumann Problem for the Upper Half-Plane	$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln(y^2+(\xi-x)^2)\,f(\xi)d\xi+c$