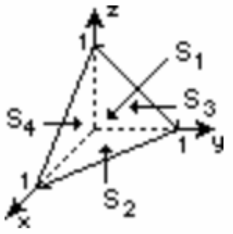
Sec 9.16

2. div $\mathbf{F} = 6y + 4z$



The Triple Integral:

$$\iiint_D \text{div } \mathbf{F}\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (6y + 4z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (6yz + 2z^2) \Big|_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (-4y^2 + 2y - 2xy + 2x^2 - 4x + 2)\, dy\, dx$$

$$= \int_0^1 \left(-\frac{4}{3}y^3 + y^2 - xy^2 + 2x^2 y - 4xy + 2y\right) \Big|_0^{1-x} dx$$

$$= \int_0^1 \left(-\frac{5}{3}x^3 + 5x^2 - 5x + \frac{5}{3}\right) dx = \left(-\frac{5}{12}x^4 + \frac{5}{3}x^3 - \frac{5}{2}x^2 + \frac{5}{3}x\right) \Big|_0^1 = \frac{5}{12}$$

The Surface Integral: Let the surfaces be S_1 in the plane $x + y + z = 1$, S_2 in $z = 0$, S_3 in $x = 0$, and S_4 in $y = 0$. The unit outward normal vectors are $\mathbf{n}_1 = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, $\mathbf{n}_2 = -\mathbf{k}$, $\mathbf{n}_3 = -\mathbf{i}$, and $\mathbf{n}_4 = -\mathbf{j}$, respectively. Now on S_1, $dS_1 = \sqrt{3}\, dA_1$, on S_3, $x = 0$, and on S_4, $y = 0$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS_1 + \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS_2 + \iint_{S_3} \mathbf{F} \cdot (-\mathbf{j})\, dS_3 + \iint_{S_4} \mathbf{F} \cdot (-\mathbf{i})\, dS_4$$

$$= \int_0^1 \int_0^{1-x} (6xy + 4y(1 - x - y) + xe^{-y})\, dy\, dx + \int_0^1 \int_0^{1-x} (-xe^{-y})\, dy\, dx$$

$$+ \iint_{S_3} (-6xy)\, dS_3 + \iint_{S_4} (-4yz)\, dS_4$$

$$= \int_0^1 \left(xy^2 + 2y^2 - \frac{4}{3}y^3 - xe^{-y}\right) \Big|_0^{1-x} dx + \int_0^1 xe^{-y} \Big|_0^{1-x} dx + 0 + 0$$

$$= \int_0^1 [x(1-x)^2 + 2(1-x)^2 - \frac{4}{3}(1-x)^3 - xe^{x-1} + x]\, dx + \int_0^1 (xe^{x-1} - x)\, dx$$

$$= \left[\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - \frac{2}{3}(1-x)^3 + \frac{1}{3}(1-x)^4\right] \Big|_0^1 = \frac{5}{12}.$$

4. div $\mathbf{F} = 4 + 1 + 4 = 9$. Using the formula for the volume of a sphere,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D 9\, dV = 9\left(\frac{4}{3}\pi 2^3\right) = 96\pi.$$

11. div $\mathbf{F} = 2z + 10y - 2z = 10y$.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 10y\,dV = \int_0^2 \int_0^{2-x^2/2} \int_z^{4-z} 10y\,dy\,dz\,dx = \int_0^2 \int_0^{2-x^2/2} 5y^2 \Big|_z^{4-z} dz\,dx$$

$$= \int_0^2 \int_0^{2-x^2/2} (80 - 40z)\,dz\,dx = \int_0^2 (80z - 20z^2) \Big|_0^{2-x^2/2} dx = \int_0^2 (80 - 5x^4)\,dx = (80x - x^5) \Big|_0^2 = 128$$