Sec 15.4

6. Since the domain of x is $(0,\infty)$ and the condition at $x=0$ involves $\partial u/\partial x$ we use the Fourier cosine transform:

$$-k\alpha^2 U(\alpha,t) - ku_x(0,t) = \frac{dU}{dt}$$

$$\frac{dU}{dt} + k\alpha^2 U = kA$$

$$U(\alpha,t) = ce^{-k\alpha^2 t} + \frac{A}{\alpha^2}.$$

Since

$$\mathscr{F}\{u(x,0)\} = U(\alpha,0) = 0$$

we find $c = -A/\alpha^2$, so that

$$U(\alpha,t) = A\frac{1-e^{-k\alpha^2 t}}{\alpha^2}.$$

Applying the inverse Fourier cosine transform we obtain

$$u(x,t) = \mathscr{F}_C^{-1}\{U(\alpha,t)\} = \frac{2A}{\pi}\int_0^\infty \frac{1-e^{-k\alpha^2 t}}{\alpha^2}\cos\alpha x\,d\alpha.$$

10. Using the Fourier sine transform we obtain

$$U(\alpha, t) = c_1 \cos \alpha a t + c_2 \sin \alpha a t.$$

Now

$$\mathscr{F}_S\{u(x,0)\} = \mathscr{F}_S\left\{xe^{-x}\right\} = \int_0^\infty xe^{-x} \sin \alpha x \, dx = \frac{2\alpha}{(1+\alpha^2)^2} = U(\alpha, 0).$$

Also,

$$\mathscr{F}_S\{u_t(x,0)\} = \left.\frac{dU}{dt}\right|_{t=0} = 0.$$

This last condition gives $c_2 = 0$. Then $U(\alpha, 0) = 2\alpha/(1+\alpha^2)^2$ yields $c_1 = 2\alpha/(1+\alpha^2)^2$. Therefore

$$U(\alpha, t) = \frac{2\alpha}{(1+\alpha^2)^2} \cos \alpha a t$$

and

$$u(x,t) = \frac{4}{\pi} \int_0^\infty \frac{\alpha \cos \alpha a t}{(1+\alpha^2)^2} \sin \alpha x \, d\alpha.$$

12. Since the boundary condition at $y = 0$ now involves $u(x, 0)$ rather than $u'(x, 0)$, we use the Fourier sine transform. The transform of the partial differential equation is then

$$\frac{d^2U}{dx^2} - \alpha^2 U + \alpha u(x,0) = 0 \qquad \text{or} \qquad \frac{d^2U}{dx^2} - \alpha^2 U = -\alpha.$$

The solution of this differential equation is

$$U(x,\alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x + \frac{1}{\alpha}.$$

The transforms of the boundary conditions at $x = 0$ and $x = \pi$ in turn imply that $c_1 = 1/\alpha$ and

$$c_2 = \frac{\cosh \alpha\pi}{\alpha \sinh \alpha\pi} - \frac{1}{\alpha \sinh \alpha\pi} + \frac{\alpha}{(1+\alpha^2)\sinh \alpha\pi}.$$

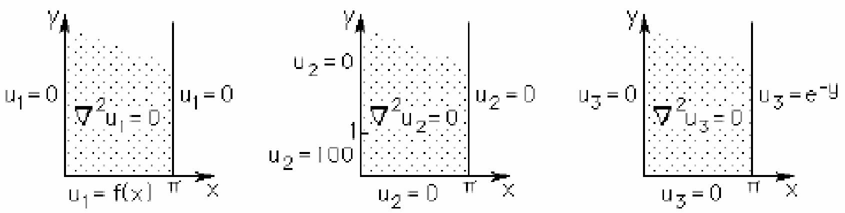
Hence

$$\begin{aligned} U(x,\alpha) &= \frac{1}{\alpha} - \frac{\cosh \alpha x}{\alpha} + \frac{\cosh \alpha\pi}{\alpha \sinh \alpha\pi}\sinh \alpha x - \frac{\sinh \alpha x}{\alpha \sinh \alpha\pi} + \frac{\alpha \sinh \alpha x}{(1+\alpha^2)\sinh \alpha\pi} \\ &= \frac{1}{\alpha} - \frac{\sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi} - \frac{\sinh \alpha x}{\alpha(1+\alpha^2)\sinh \alpha\pi}. \end{aligned}$$

Taking the inverse transform it follows that

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{\alpha} - \frac{\sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi} - \frac{\sinh \alpha x}{\alpha(1+\alpha^2)\sinh \alpha\pi}\right)\sin \alpha y \, d\alpha.$$

18. We solve the three boundary-value problems:



Using separation of variables we find the solution of the first problem is

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx \quad \text{where} \quad A_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx.$$

Using the Fourier sine transform with respect to y gives the solution of the second problem:

$$u_2(x,y) = \frac{200}{\pi}\int_0^\infty \frac{(1 - \cos \alpha)\sinh \alpha(\pi - x)}{\alpha \sinh \alpha\pi}\sin \alpha y \, d\alpha.$$

Also, the Fourier sine transform with respect to y gives the solution of the third problem:

$$u_3(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sinh \alpha x}{(1+\alpha^2)\sinh \alpha\pi}\sin \alpha y \, d\alpha.$$

The solution of the original problem is

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y).$$