

Sec 15.2

6. Transforming the partial differential equation gives

$$\frac{d^2U}{dx^2} - s^2U = -\frac{\omega}{s^2 + \omega^2} \sin \pi x.$$

Using undetermined coefficients we obtain

$$U(x, s) = c_1 \cosh sx + c_2 \sinh sx + \frac{\omega}{(s^2 + \pi^2)(s^2 + \omega^2)} \sin \pi x.$$

The transformed boundary conditions $U(0, s) = 0$ and $U(1, s) = 0$ give, in turn, $c_1 = 0$ and $c_2 = 0$. Therefore

$$U(x, s) = \frac{\omega}{(s^2 + \pi^2)(s^2 + \omega^2)} \sin \pi x$$

and

$$\begin{aligned} u(x, t) &= \omega \sin \pi x \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + \pi^2)(s^2 + \omega^2)} \right\} \\ &= \frac{\omega}{\omega^2 - \pi^2} \sin \pi x \mathcal{L}^{-1} \left\{ \frac{1}{\pi} \frac{\pi}{s^2 + \pi^2} - \frac{1}{\omega} \frac{\omega}{s^2 + \omega^2} \right\} \\ &= \frac{\omega}{\pi(\omega^2 - \pi^2)} \sin \pi t \sin \pi x - \frac{1}{\omega^2 - \pi^2} \sin \omega t \sin \pi x. \end{aligned}$$

8. We use

$$U(x, s) = c_1 e^{-(x/a)s} + c_2 e^{(x/a)s} - \frac{v_0}{s^2}.$$

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Now $\lim_{x \rightarrow \infty} dU/dx = 0$ implies $c_2 = 0$, and $U(0, s) = 0$ then gives $c_1 = v_0/s^2$. Hence

$$U(x, s) = \frac{v_0}{s^2} e^{-(x/a)s} - \frac{v_0}{s^2}$$

and

$$u(x, t) = v_0 \left(t - \frac{x}{a} \right) \mathcal{U} \left(t - \frac{x}{a} \right) - v_0 t.$$

10. We use

$$U(x, s) = c_1 e^{-xs} + c_2 e^{xs} + \frac{s}{s^2 - 1} e^{-x}.$$

Now $\lim_{x \rightarrow \infty} u(x, t) = 0$ implies $\lim_{x \rightarrow \infty} U(x, s) = 0$, so we define $c_2 = 0$. Then

$$U(x, s) = c_1 e^{-xs} + \frac{s}{s^2 - 1} e^{-x}.$$

Finally, $U(0, s) = 1/s$ gives $c_1 = 1/s - s/(s^2 - 1)$. Thus

$$U(x, s) = \frac{1}{s} - \frac{s}{s^2 - 1} e^{-xs} + \frac{s}{s^2 - 1} e^{-x}$$

and

$$\begin{aligned} u(x, t) &= -\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - 1} e^{-(x/a)s} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2 - 1} \right\} e^{-x} \\ &= -\cosh \left(t - \frac{x}{a} \right) \mathcal{U} \left(t - \frac{x}{a} \right) + e^{-x} \cosh t. \end{aligned}$$

14. We use

$$U(x, s) = c_1 e^{-\sqrt{s}x} + c_2 e^{\sqrt{s}x}.$$

The condition $\lim_{x \rightarrow \infty} u(x, t) = 0$ implies $\lim_{x \rightarrow \infty} U(x, s) = 0$, so we define $c_2 = 0$. Hence

$$U(x, s) = c_1 e^{-\sqrt{s}x}.$$

The remaining boundary condition transforms into

$$\left. \frac{dU}{dx} \right|_{x=0} = U(0, s) - \frac{50}{s}.$$

This condition gives $c_1 = 50/s(\sqrt{s} + 1)$. Therefore

$$U(x, s) = 50 \frac{e^{-\sqrt{s}x}}{s(\sqrt{s} + 1)}$$

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and

$$u(x, t) = 50 \mathcal{L}^{-1} \left\{ \frac{e^{-x\sqrt{s}}}{s(\sqrt{s} + 1)} \right\} = -50e^{x+2t} \operatorname{erfc} \left(\sqrt{t} + \frac{x}{2\sqrt{t}} \right) + 50 \operatorname{erfc} \left(\frac{x}{2\sqrt{t}} \right).$$

28. (a) We use

$$U(x, s) = c_1 e^{-\sqrt{s/k}x} + c_2 e^{\sqrt{s/k}x}.$$

Now $\lim_{x \rightarrow \infty} u(x, t) = 0$ implies $\lim_{x \rightarrow \infty} U(x, s) = 0$, so we define $c_2 = 0$. Then

$$U(x, s) = c_1 e^{-\sqrt{s/k}x}.$$

Finally, from $U(0, s) = u_0/s$ we obtain $c_1 = u_0/s$. Thus

$$U(x, s) = u_0 \frac{e^{-\sqrt{s/k}x}}{s}$$

and

$$u(x, t) = u_0 \mathcal{L}^{-1} \left\{ \frac{e^{-\sqrt{s/k}x}}{s} \right\} = u_0 \mathcal{L}^{-1} \left\{ \frac{e^{-(x/\sqrt{k})\sqrt{s}}}{s} \right\} = u_0 \operatorname{erfc} \left(\frac{x}{2\sqrt{kt}} \right).$$

Since $\operatorname{erfc}(0) = 1$,

$$\lim_{t \rightarrow \infty} u(x, t) = \lim_{t \rightarrow \infty} u_0 \operatorname{erfc}(x/2\sqrt{kt}) = u_0.$$

(b)

