

Sec 14.3

4. The coefficients are given by

$$A_n = \frac{2n+1}{2c^n} \int_0^\pi (1 - \cos 2\theta) P_n(\cos \theta) \sin \theta \, d\theta.$$

These were computed in Problem 18 of Section 12.6. Thus

$$u(r, \theta) = \frac{4}{3} P_0(\cos \theta) - \frac{4}{3} \left(\frac{r}{c}\right)^2 P_2(\cos \theta).$$

6. Referring to Example 1 in the text we have

$$R(r) = c_1 r^n \quad \text{and} \quad \Theta(\theta) = P_n(\cos \theta).$$

Now $\Theta(\pi/2) = 0$ implies that n is odd, so

$$u(r, \theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1} P_{2n+1}(\cos \theta).$$

From

$$u(c, \theta) = f(\theta) = \sum_{n=0}^{\infty} A_{2n+1} c^{2n+1} P_{2n+1}(\cos \theta)$$

we see that

$$A_{2n+1} c^{2n+1} = (4n+3) \int_0^{\pi/2} f(\theta) \sin \theta P_{2n+1}(\cos \theta) \, d\theta.$$

Thus

$$u(r, \theta) = \sum_{n=0}^{\infty} A_{2n+1} r^{2n+1} P_{2n+1}(\cos \theta)$$

where

$$A_{2n+1} = \frac{4n+3}{c^{2n+1}} \int_0^{\pi/2} f(\theta) \sin \theta P_{2n+1}(\cos \theta) \, d\theta.$$

12. Proceeding as in Example 1 we obtain

$$\Theta(\theta) = P_n(\cos \theta) \quad \text{and} \quad R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

so that

$$u(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos \theta).$$

To satisfy $\lim_{r \rightarrow \infty} u(r, \theta) = -Er \cos \theta$ we must have $A_n = 0$ for $n = 2, 3, 4, \dots$. Then

$$\lim_{r \rightarrow \infty} u(r, \theta) = -Er \cos \theta = A_0 \cdot 1 + A_1 r \cos \theta,$$

so $A_0 = 0$ and $A_1 = -E$. Thus

$$u(r, \theta) = -Er \cos \theta + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos \theta).$$

Now

$$u(c, \theta) = 0 = -Ec \cos \theta + \sum_{n=0}^{\infty} B_n c^{-(n+1)} P_n(\cos \theta)$$

so

$$\sum_{n=0}^{\infty} B_n c^{-(n+1)} P_n(\cos \theta) = Ec \cos \theta$$

and

$$B_n c^{-(n+1)} = \frac{2n+1}{2} \int_0^\pi Ec \cos \theta P_n(\cos \theta) \sin \theta d\theta.$$

Now $\cos \theta = P_1(\cos \theta)$ so, for $n \neq 1$,

$$\int_0^\pi \cos \theta P_n(\cos \theta) \sin \theta d\theta = 0$$

by orthogonality. Thus $B_n = 0$ for $n \neq 1$ and

$$B_1 = \frac{3}{2} Ec^3 \int_0^\pi \cos^2 \theta \sin \theta d\theta = Ec^3.$$

Therefore,

$$u(r, \theta) = -Er \cos \theta + Ec^3 r^{-2} \cos \theta.$$