Sec 13.5

2. Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$
$$Y'(0) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_3 \cosh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \, \cosh \frac{n\pi}{a} y.$$

Imposing

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi b}{a} \, \sin \frac{n\pi}{a} x$$

gives

$$A_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

so that

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \, \cosh \frac{n\pi}{a} y$$

where

$$A_n = \frac{2}{a} \operatorname{sech} \frac{n\pi b}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx.$$

4. Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X'(0) = 0,$$
$$X'(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$
$$Y(b) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_1 \cos \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_3 \cosh \frac{n\pi}{a} y - c_3 \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$. Since $\lambda = 0$ is an eigenvalue for both differential equations with corresponding eigenfunctions 1 and $y - b$, respectively we have

$$u = A_0(y-b) + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a} x \left(\cosh \frac{n\pi}{a} y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y \right).$$

Imposing

$$u(x,0) = x = -A_0 b + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a} x$$

gives

$$-A_0 b = \frac{1}{a} \int_0^a x\, dx = \frac{1}{2} a$$

and

$$A_n = \frac{2}{a} \int_0^a x \cos \frac{n\pi}{a} x\, dx = \frac{2a}{n^2\pi^2}[(-1)^n - 1]$$

so that

$$u(x,y) = \frac{a}{2b}(b-y) + \frac{2a}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi}{a} x \left(\cosh \frac{n\pi}{a} y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y \right).$$

8. Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) &= 0, \\ X(1) &= 0, \end{aligned}$$

and

$$\begin{aligned} Y'' - \lambda Y &= 0, \\ Y'(0) &= Y(0). \end{aligned}$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin n\pi x \qquad \text{and} \qquad Y = c_4(n \cosh n\pi y + \sinh n\pi y)$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n (n \cosh n\pi y + \sinh n\pi y) \sin n\pi x.$$

Imposing

$$u(x,1) = f(x) = \sum_{n=1}^{\infty} A_n (n \cosh n\pi + \sinh n\pi) \sin n\pi x$$

gives

$$A_n (n \cosh n\pi + \sinh n\pi) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin n\pi x \, dx$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,y) = \sum_{n=1}^{\infty} A_n (n \cosh n\pi y + \sinh n\pi y) \sin n\pi x$$

where

$$A_n = \frac{2}{n\pi \cosh n\pi + \pi \sinh n\pi} \int_0^1 f(x) \sin n\pi x \, dx.$$

10. This boundary-value problem has the form of Problem 2 in this section, with $a = 1$ and $b = 1$. Thus, the solution has the form

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi x + B_n \sinh n\pi x) \sin n\pi y.$$

The boundary condition $u(0, y) = 10y$ implies

$$10y = \sum_{n=1}^{\infty} A_n \sin n\pi y$$

and

$$A_n = \frac{2}{1} \int_0^1 10y \sin n\pi y \, dy = \frac{20}{n\pi}(-1)^{n+1}.$$

At $x = a$,

$$G(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a \right) \sin \frac{n\pi}{b} y$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy.$$

We can now solve for B_n:

$$B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a} \left(\frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a \right). \qquad \textbf{(6)}$$

A solution to the given boundary-value problem consists of the series (**4**) with coefficients A_n and B_n given in (**5**) and (**6**), respectively.

At $x = a$,

$$G(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a \right) \sin \frac{n\pi}{b} y$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy.$$

We can now solve for B_n:

$$B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a} \left(\frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a \right). \qquad \textbf{(6)}$$

A solution to the given boundary-value problem consists of the series (**4**) with coefficients A_n and B_n given in (**5**) and (**6**), respectively.

14. Since the boundary conditions at $x = 0$ and $x = a$ are functions of y we choose to separate Laplace's equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

so that

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0.$$

Then with $\lambda = -\alpha^2$ we have

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$Y(y) = c_3 \cos \alpha y + c_4 \sin \alpha y.$$

Now $Y(0) = 0$ gives $c_3 = 0$ and $Y(b) = 0$ implies $\sin \alpha b = 0$ or $\alpha = n\pi/b$ for $n = 1, 2, 3, \ldots$. Thus

$$u_n(x, y) = XY = \left(A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x\right) \sin \frac{n\pi}{b} y$$

and

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x\right) \sin \frac{n\pi}{b} y. \qquad (4)$$

At $x = 0$ we then have

$$F(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b} y$$

and consequently

$$A_n = \frac{2}{b} \int_0^b F(y) \sin \frac{n\pi}{b} y \, dy. \qquad (5)$$