Sec 13.4

2. Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T'' + \lambda a^2 T = 0,$$
$$T(0) = 0.$$

Solving the differential equations we get

$$X = c_1 \sin \frac{n\pi}{L} x + c_2 \cos \frac{n\pi}{L} x \qquad \text{and} \qquad T = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t$$

for $n = 1, 2, 3, \ldots$. The boundary and initial conditions give

$$u = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x.$$

Imposing

$$u_t(x, 0) = x(L - x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x$$

gives

$$B_n \frac{n\pi a}{L} = \frac{4L^2}{n^3 \pi^3} [1 - (-1)^n]$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{4L^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^4} \sin \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x.$$

4. Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(\pi) = 0,$$

and

$$T'' + \lambda a^2 T = 0,$$
$$T'(0) = 0.$$

Solving the differential equations we get

$$X = c_1 \sin nx + c_2 \cos nx \qquad \text{and} \qquad T = c_3 \cos nat + c_4 \sin nat$$

for $n = 1, 2, 3, \ldots$. The boundary and initial conditions give

$$u = \sum_{n=1}^{\infty} A_n \cos nt \, \sin nx.$$

Imposing

$$u(x,0) = \frac{1}{6}x(\pi^2 - x^2) = \sum_{n=1}^{\infty} A_n \sin nx \qquad \text{and} \qquad u_t(x,0) = 0$$

gives

$$A_n = \frac{2}{n^3}(-1)^{n+1}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \cos nat \, \sin nx.$$

6. Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0,$$

and

$$T'' + \lambda a^2 T = 0,$$
$$T'(0) = 0.$$

Solving the differential equations we get

$$X = c_1 \sin n\pi x + c_2 \cos n\pi x \qquad \text{and} \qquad T = c_3 \cos n\pi at + c_4 \sin n\pi at$$

for $n = 1, 2, 3, \ldots$. The boundary and initial conditions give

$$u = \sum_{n=1}^{\infty} A_n \cos nt \, \sin nx.$$

Imposing

$$u(x,0) = 0.01 \sin 3\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

gives $A_3 = 0.01$, and $A_n = 0$ for $n = 1, 2, 4, 5, 6, \ldots$ so that

$$u(x,t) = 0.01 \sin 3\pi x \, \cos 3\pi at.$$

8. Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X'(0) = 0,$$
$$X'(L) = 0,$$

and

$$T'' + \lambda a^2 T = 0,$$
$$T'(0) = 0.$$

Solving the differential equations we get

$$X = c_1 \sin \frac{n\pi}{L} x + c_2 \cos \frac{n\pi}{L} x \qquad \text{and} \qquad T = c_3 \cos \frac{n\pi a}{L} t + c_4 \sin \frac{n\pi a}{L} t$$

for $n = 1, 2, 3, \ldots$. The boundary and initial conditions, together with the fact that $\lambda = 0$ is an eigenvalue with eigenfunction $X(x) = 1$, give

$$u = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x.$$

Imposing

$$u(x, 0) = x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{L} x$$

gives

$$A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{L}{2}$$

and

$$A_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi}{L} x \, dx = \frac{2L}{n^2 \pi^2} [(-1)^n - 1]$$

for $n = 1, 2, 3, \ldots$, so that

$$u(x, t) = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi a}{L} t \cos \frac{n\pi}{L} x.$$

10. Using $u = XT$ and $-\lambda =$ as a separation constant leads to $X'' + \alpha^2 X = 0$, $X(0) = 0$, $X(\pi) = 0$ and $T'' + (1+\alpha^2)T = 0$, $T'(0) = 0$. Then $X = c_2 \sin nx$ and $T = c_3 \cos \sqrt{n^2+1}\,t$ for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n1}^{\infty} B_n \cos \sqrt{n^2+1}\,t \sin nx.$$

Imposing $u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$ gives

$$B_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & n \text{ even,} \\ \dfrac{4}{\pi n^2}(-1)^{(n+3)/2}, & n = 2k-1,\ k = 1, 2, 3, \ldots. \end{cases}$$

Thus with $n = 2k - 1$,

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2}\cos\sqrt{n^2+1}\,t \sin nx = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\cos\sqrt{(2k-1)^2+1}\,t\sin(2k-1)x.$$