Sec 13.3

2. Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(L) = 0,$$

and

$$T' + k\lambda T = 0.$$

This leads to

$$X = c_1 \sin\frac{n\pi}{L}x \qquad \text{and} \qquad T = c_2 e^{-kn^2\pi^2 t/L^2}$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \sin\frac{n\pi}{L}x.$$

Imposing

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

gives

$$A_n = \frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi}{L}x\,dx = \frac{4L^2}{n^3\pi^3}[1-(-1)^n]$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,t) = \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}e^{-kn^2\pi^2 t/L^2}\sin\frac{n\pi}{L}x.$$

4. If $L = 2$ and $f(x)$ is x for $0 < x < 1$ and $f(x)$ is 0 for $1 < x < 2$ then

$$u(x,t) = \frac{1}{4} + 4\sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\sin\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right]e^{-kn^2\pi^2 t/4}\cos\frac{n\pi}{2}x.$$

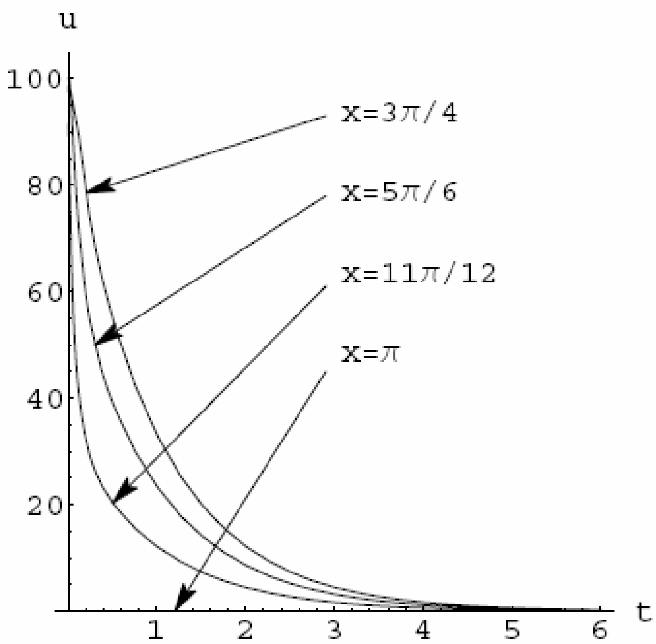
6. In Problem 5 we instead find that $X(0) = 0$ and $X(L) = 0$ so that

$$X = c_1 \sin \frac{n\pi}{L} x$$

and

$$u = \frac{2e^{-ht}}{L} \sum_{n=1}^{\infty} \left(\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \right) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$

8.



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Imposing

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

gives

$$A_n = \frac{2}{L} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx = \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right)$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2}}{n} \, e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$