Sec 13.1

2. Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X'Y + 3XY' = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{-3X} = \frac{Y'}{Y} = -\lambda.$$

When $\lambda \neq 0$

$$X' - 3\lambda X = 0 \qquad \text{and} \qquad Y' + \lambda Y = 0$$

so that

$$X = c_1 e^{3\lambda x} \qquad \text{and} \qquad Y = c_2 e^{-\lambda y}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{\lambda(3x-y)}.$$

When $\lambda = 0$ the differential equations become $X' = 0$ and $Y' = 0$, so in this case $X = c_4$, $Y = c_5$, and $u = XY = c_6$.

10. Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $kX''T = XT'$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda.$$

Then

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + \lambda kT = 0.$$

The second differential equation implies $T(t) = c_1 e^{-\lambda kt}$. For the first differential equation we consider three cases:

I. If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_2 x + c_3$, so

$$u = XT = A_1 x + A_2.$$

II. If $\lambda = -\alpha^2 < 0$, then $X'' - \alpha^2 X = 0$, and $X(x) = c_4 \cosh \alpha x + c_5 \sinh \alpha x$, so

$$u = XT = (A_3 \cosh \alpha x + A_4 \sinh \alpha x) e^{k\alpha^2 t}.$$

III. If $\lambda = \alpha^2 > 0$, then $X'' + \alpha^2 X = 0$, and $X(x) = c_6 \cos \alpha x + c_7 \sin \alpha x$, so

$$u = XT = (A_5 \cos \alpha x + A_6 \sin \alpha x) e^{-k\alpha^2 t}.$$

14. Substituting $u(x,y)=X(x)Y(y)$ into the partial differential equation yields $x^2X''Y+XY''=0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$-\frac{x^2X''}{X}=\frac{Y''}{Y}=-\lambda.$$

Then

$$x^2X''-\lambda X=0 \qquad \text{and} \qquad Y''+\lambda Y=0.$$

We consider three cases:

I. If $\lambda=0$ then $x^2X''=0$ and $X(x)=c_1x+c_2$. Also, $Y''=0$ and $Y(y)=c_3y+c_4$ so

$$u=XY=(c_1x+c_2)(c_3y+c_4).$$

II. If $\lambda=-\alpha^2<0$ then $x^2X''+\alpha^2X=0$ and $Y''-\alpha^2Y=0$. The solution of the second differential equation is $Y(y)=c_5\cosh\alpha y+c_6\sinh\alpha y$. The first equation is Cauchy-Euler with auxiliary equation $m^2-m+\alpha^2=0$. Solving for m we obtain $m=\frac{1}{2}\pm\frac{1}{2}\sqrt{1-4\alpha^2}$. We consider three possibilities for the discriminant $1-4\alpha^2$.

(i) If $1-4\alpha^2=0$ then $X(x)=c_7x^{1/2}+c_8x^{1/2}\ln x$ and

$$u=XY=x^{1/2}(c_7+c_8\ln x)(c_5\cosh\alpha y+c_6\sinh\alpha y).$$

(ii) If $1-4\alpha^2<0$ then $X(x)=x^{1/2}\left[c_9\cos\left(\sqrt{4\alpha^2-1}\,\ln x\right)+c_{10}\sin\left(\sqrt{4\alpha^2-1}\,\ln x\right)\right]$ and

$$u=XY=x^{1/2}\left[c_9\cos\left(\sqrt{4\alpha^2-1}\,\ln x\right)+c_{10}\sin\left(\sqrt{4\alpha^2-1}\,\ln x\right)\right](c_5\cosh\alpha y+c_6\sinh\alpha y).$$

(iii) If $1-4\alpha^2>0$ then $X(x)=x^{1/2}\left(c_{11}x^{\sqrt{1-4\alpha^2}/2}+c_{12}x^{-\sqrt{1-4\alpha^2}/2}\right)$ and

$$u=XY=x^{1/2}\left(c_{11}x^{\sqrt{1-4\alpha^2}/2}+c_{12}x^{-\sqrt{1-4\alpha^2}/2}\right)(c_5\cosh\alpha y+c_6\sinh\alpha y).$$

III. If $\lambda=\alpha^2>0$ then $x^2X''-\alpha^2X=0$ and $Y''+\alpha^2Y=0$. The solution of the second differential equation is $Y(y)=c_{13}\cos\alpha y+c_{14}\sin\alpha y$. The first equation is Cauchy-Euler with auxiliary equation $m^2-m-\alpha^2=0$. Solving for m we obtain $m=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4\alpha^2}$. In this case the discriminant is always positive so the solution of the differential equation is $X(x)=x^{1/2}\left(c_{15}x^{\sqrt{1+4\alpha^2}/2}+c_{16}x^{-\sqrt{1+4\alpha^2}/2}\right)$ and

$$u=XY=x^{1/2}\left(c_{15}x^{\sqrt{1+4\alpha^2}/2}+c_{16}x^{-\sqrt{1+4\alpha^2}/2}\right)(c_{13}\cos\alpha y+c_{14}\sin\alpha y).$$

18. Identifying $A=3$, $B=5$, and $C=1$, we compute $B^2-4AC=13>0$. The equation is hyperbolic.

20. Identifying $A=1$, $B=-1$, and $C=-3$, we compute $B^2-4AC=13>0$. The equation is hyperbolic.

24. Identifying $A=1$, $B=0$, and $C=1$, we compute $B^2-4AC=-4<0$. The equation is elliptic.

28. Substituting $u(r,\theta) = R(r)\Theta(\theta)$ into the partial differential equation yields

$$R''\Theta + \frac{1}{r}\,R'\Theta + \frac{1}{r^2}\,R\Theta'' = 0.$$

Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\lambda.$$

Then

$$r^2R'' + rR' + \lambda R = 0 \qquad \text{and} \qquad \Theta'' - \lambda\Theta = 0.$$

Letting $\lambda = -\alpha^2$ we have the Cauchy-Euler equation $r^2R'' + rR' - \alpha^2 R = 0$ whose solution is $R(r) = c_3 r^{\alpha} + c_4 r^{-\alpha}$. Since the solution of $\Theta'' + \alpha^2\Theta = 0$ is $\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta$ we see that a solution of the partial differential equation is

$$u = R\Theta = (c_1 \cos\alpha\theta + c_2 \sin\alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha}).$$