Sec 12.6

2. By (6) in the text $J_0'(2\alpha) = -J_1(2\alpha)$. Thus, $J_0'(2\alpha) = 0$ is equivalent to $J_1(2\alpha)$. Then $\alpha_1 = 1.9159$, $\alpha_2 = 3.5078$, $\alpha_3 = 5.0867$, and $\alpha_4 = 6.6618$.

4. The boundary condition indicates that we use (19) and (20) in the text. With $b = 2$ we obtain

$$c_1 = \frac{2}{4}\int_0^2 x\,dx = \frac{2}{4}\frac{x^2}{2}\Big|_0^2 = 1,$$

$$\begin{aligned} c_i &= \frac{2}{4J_0^2(2\alpha_i)}\int_0^2 xJ_0(\alpha_i x)\,dx && \boxed{t = \alpha_i x \qquad dt = \alpha_i\,dx} \\ &= \frac{1}{2J_0^2(2\alpha_i)}\cdot\frac{1}{\alpha_i^2}\int_0^{2\alpha_i} tJ_0(t)\,dt \\ &= \frac{1}{2\alpha_i^2 J_0^2(2\alpha_i)}\int_0^{2\alpha_i}\frac{d}{dt}[tJ_1(t)]\,dt && \text{[From (5) in the text]} \\ &= \frac{1}{2\alpha_i^2 J_0^2(2\alpha_i)}tJ_1(t)\Big|_0^{2\alpha_i} \\ &= \frac{J_1(2\alpha_i)}{\alpha_i J_0^2(2\alpha_i)}. \end{aligned}$$

Now since $J_0'(2\alpha_i) = 0$ is equivalent to $J_1(2\alpha_i) = 0$ we conclude $c_i = 0$ for $i = 2, 3, 4, \ldots$. Thus the expansion of f on $0 < x < 2$ consists of a series with one nontrivial term:

$$f(x) = c_1 = 1.$$

6. Writing the boundary condition in the form

$$2J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$$

we identify $b = 2$ and $h = 2$. Using (17) and (18) in the text we obtain

$$\boxed{t = \alpha_i x \qquad dt = \alpha_i\, dx}$$

$$\begin{aligned}
c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2 + 4)J_0^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x)\, dx \\
&= \frac{\alpha_i^2}{2(\alpha_i^2 + 1)J_0^2(2\alpha_i)} \cdot \frac{1}{\alpha_i^2} \int_0^{2\alpha_i} t J_0(t)\, dt \\
&= \frac{1}{2(\alpha_i^2 + 1)J_0^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{d}{dt}[tJ_1(t)]\, dt && \text{[From (5) in the text]} \\
&= \frac{1}{2(\alpha_i^2 + 1)J_0^2(2\alpha_i)} tJ_1(t)\Big|_0^{2\alpha_i} \\
&= \frac{\alpha_i J_1(2\alpha_i)}{(\alpha_i^2 + 1)J_0^2(2\alpha_i)}\,.
\end{aligned}$$

Thus

$$f(x) = \sum_{i=1}^{\infty} \frac{\alpha_i J_1(2\alpha_i)}{(\alpha_i^2 + 1)J_0^2(2\alpha_i)} J_0(\alpha_i x).$$

8. The boundary condition indicates that we use (15) and (16) in the text. With $n = 2$ and $b = 1$ we obtain

$$\boxed{t = \alpha_i x \qquad dt = \alpha_i\, dx}$$

$$\begin{aligned}
c_1 &= \frac{2}{J_3^2(\alpha_i)} \int_0^1 x J_2(\alpha_i x) x^2\, dx \\
&= \frac{2}{J_3^2(\alpha_i)} \cdot \frac{1}{\alpha_i^4} \int_0^{\alpha_i} t^3 J_2(t)\, dt \\
&= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} \int_0^{\alpha_i} \frac{d}{dt}[t^3 J_3(t)]\, dt && \text{[From (5) in the text]} \\
&= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} t^3 J_3(t)\Big|_0^{\alpha_i} \\
&= \frac{2}{\alpha_i J_3(\alpha_i)}\,.
\end{aligned}$$

Thus

$$f(x) = 2\sum_{i=1}^{\infty} \frac{1}{\alpha_i J_3(\alpha_i)} J_2(\alpha_i x).$$

10. The boundary condition indicates that we use (15) and (16) in the text. With $b = 1$ it follows that

$$c_i = \frac{2}{J_1^2(\alpha_i)} \int_0^1 x\left(1 - x^2\right) J_0(\alpha_i x)\, dx$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\int_0^1 x J_0(\alpha_i x)\, dx - \int_0^1 x^3 J_0(\alpha_i x)\, dx \right]$$

$$\boxed{t = \alpha_i x \qquad dt = \alpha_i\, dx}$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2} \int_0^{\alpha_i} t J_0(t)\, dt - \frac{1}{\alpha_i^4} \int_0^{\alpha_i} t^3 J_0(t)\, dt \right]$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2} \int_0^{\alpha_i} \frac{d}{dt}\left[t J_1(t) \right] dt - \frac{1}{\alpha_i^4} \int_0^{\alpha_i} t^2 \frac{d}{dt}\left[t J_1(t) \right] dt \right]$$

$$\boxed{\begin{array}{ll} u = t^2 & dv = \frac{d}{dt}\left[t J_1(t) \right] dt \\ du = 2t\, dt & v = t J_1(t) \end{array}}$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2} t J_1(t) \Big|_0^{\alpha_i} - \frac{1}{\alpha_i^4} \left(t^3 J_1(t) \Big|_0^{\alpha_i} - 2 \int_0^{\alpha_i} t^2 J_1(t)\, dt \right) \right]$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\frac{J_1(\alpha_i)}{\alpha_i} - \frac{J_1(\alpha_i)}{\alpha_i} + \frac{2}{\alpha_i^4} \int_0^{\alpha_i} \frac{d}{dt}\left[t^2 J_2(t) \right] dt \right]$$

$$= \frac{2}{J_1^2(\alpha_i)} \left[\frac{2}{\alpha_i^4} t^2 J_2(t) \Big|_0^{\alpha_i} \right] = \frac{4 J_2(\alpha_i)}{\alpha_i^2 J_1^2(\alpha_i)}\,.$$

Thus

$$f(x) = 4 \sum_{i=1}^{\infty} \frac{J_2(\alpha_i)}{\alpha_i^2 J_1^2(\alpha_i)} J_0(\alpha_i x).$$

20. If f is an odd function on $(-1, 1)$ then

$$\int_{-1}^{1} f(x)P_{2n}(x)\,dx = 0$$

and

$$\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = 2\int_{0}^{1} f(x)P_{2n+1}(x)\,dx.$$

Thus

$$\begin{aligned} c_{2n+1} = \frac{2(2n+1)+1}{2}\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx &= \frac{4n+3}{2}\left(2\int_{0}^{1} f(x)P_{2n+1}(x)\,dx\right) \\ &= (4n+3)\int_{0}^{1} f(x)P_{2n+1}(x)\,dx, \end{aligned}$$

$c_{2n} = 0$, and

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1}P_{2n+1}(x).$$