Sec 12.5

2. For $\lambda < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and the boundary conditions both imply $c_1 + c_2 = 0$. Thus, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y_0 = x - 1$.

For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

and

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x.$$

The boundary conditions imply

$$c_1 + c_2 \alpha = 0$$
$$c_1 \cos \alpha + c_2 \sin \alpha = 0$$

which gives

$$-c_2 \alpha \cos \alpha + c_2 \sin \alpha = 0 \qquad \text{or} \qquad \tan \alpha = \alpha.$$

The eigenvalues are $\lambda_n = \alpha_n^2$ where α_1, α_2, α_3, ... are the consecutive positive solutions of $\tan \alpha = \alpha$. The corresponding eigenfunctions are $\alpha \cos \alpha x - \sin \alpha x$ (obtained by taking $c_2 = -1$ in the first equation of the system.) Using a CAS we find that the first four positive eigenvalues are 20.1907, 59.6795, 118.9000, and 197.858 with corresponding eigenfunctions $4.4934 \cos 4.4934x - \sin 4.4934x$, $7.7253 \cos 7.7253x - \sin 7.7253x$, $10.9041 \cos 10.9041x - \sin 10.9041x$, and $14.0662 \cos 14.0662x - \sin 14.0662x$.

4. For $\lambda = -\alpha^2 < 0$ we have

$$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$
$$y' = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x.$$

Using the fact that $\cosh x$ is an even function and $\sinh x$ is odd we have

$$\begin{aligned} y(-L) &= c_1 \cosh(-\alpha L) + c_2 \sinh(-\alpha L) \\ &= c_1 \cosh \alpha L - c_2 \sinh \alpha L \end{aligned}$$

and

$$\begin{aligned} y'(-L) &= c_1 \alpha \sinh(-\alpha L) + c_2 \alpha \cosh(-\alpha L) \\ &= -c_1 \alpha \sinh \alpha L + c_2 \alpha \cosh \alpha L. \end{aligned}$$

The boundary conditions imply

$$c_1 \cosh \alpha L - c_2 \sinh \alpha L = c_1 \cosh \alpha L + c_2 \sinh \alpha L$$

or

$$2c_2 \sinh \alpha L = 0$$

and

$$-c_1 \alpha \sinh \alpha L + c_2 \alpha \cosh \alpha L = c_1 \alpha \sinh \alpha L + c_2 \alpha \cosh \alpha L$$

or

$$2c_1 \alpha \sinh \alpha L = 0.$$

Since $\alpha L \neq 0$, $c_1 = c_2 = 0$ and the only solution of the boundary-value problem in this case is $y = 0$.

For $\lambda = 0$ we have

$$y = c_1 x + c_2$$
$$y' = c_1.$$

From $y(-L) = y(L)$ we obtain

$$-c_1 L + c_2 = c_1 L + c_2.$$

Then $c_1 = 0$ and $y = 1$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.

For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$y' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x.$$

The first boundary condition implies

$$c_1 \cos \alpha L - c_2 \sin \alpha L = c_1 \cos \alpha L + c_2 \sin \alpha L$$

or

$$2c_2 \sin \alpha L = 0.$$

Thus, if $c_1 = 0$ and $c_2 \neq 0$,

$$\alpha L = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}\,,\ n = 1, 2, 3, \ldots\,.$$

The corresponding eigenfunctions are $\sin(n\pi x/L)$, for $n = 1,\ 2,\ 3,\ \ldots$. Similarly, the second boundary condition implies

$$2c_1\alpha \sin \alpha L = 0.$$

If $c_1 \neq 0$ and $c_2 = 0$,

$$\alpha L = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}\,,\ n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions are $\cos(n\pi x/L)$, for $n = 1,\ 2,\ 3,\ \ldots$.

6. The eigenfunctions are $\sin \alpha_n x$ where $\tan \alpha_n = -\alpha_n$. Thus

$$\begin{aligned}
\|\sin \alpha_n x\|^2 = \int_0^1 \sin^2 \alpha_n x\, dx &= \frac{1}{2}\int_0^1 (1 - \cos 2\alpha_n x)\ dx \\
&= \frac{1}{2}\left(x - \frac{1}{2\alpha_n}\sin 2\alpha_n x\right)\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin 2\alpha_n\right) \\
&= \frac{1}{2}\left[1 - \frac{1}{2\alpha_n}(2\sin\alpha_n \cos\alpha_n)\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{\alpha_n}\tan\alpha_n \cos\alpha_n \cos\alpha_n\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{\alpha_n}\left(-\alpha_n \cos^2\alpha_n\right)\right] = \frac{1}{2}\left(1 + \cos^2\alpha_n\right).
\end{aligned}$$

8. **(a)** The roots of the auxiliary equation $m^2+m+\lambda=0$ are $\frac{1}{2}(-1\pm\sqrt{1-4\lambda}\,)$. When $\lambda=0$ the general solution of the differential equation is $c_1+c_2e^{-x}$. The boundary conditions imply $c_1+c_2=0$ and $c_1+c_2e^{-2}=0$. Since the determinant of the coefficients is not 0, the only solution of this homogeneous system is $c_1=c_2=0$, in which case $y=0$. When $\lambda=\frac{1}{4}$, the general solution of the differential equation is $c_1e^{-x/2}+c_2xe^{-x/2}$. The boundary conditions imply $c_1=0$ and $c_1+2c_2=0$, so $c_1=c_2=0$ and $y=0$. Similarly, if $0<\lambda<\frac{1}{4}$, the general solution is

$$y=c_1e^{\frac{1}{2}(-1+\sqrt{1-4\lambda})x}+c_2e^{\frac{1}{2}(-1-\sqrt{1-4\lambda})x}.$$

In this case the boundary conditions again imply $c_1=c_2=0$, and so $y=0$. Now, for $\lambda>\frac{1}{4}$, the general solution of the differential equation is

$$y=c_1e^{-x/2}\cos\sqrt{4\lambda-1}\,x+c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x.$$

The condition $y(0)=0$ implies $c_1=0$ so $y=c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x$. From

$$y(2)=c_2e^{-1}\sin 2\sqrt{4\lambda-1}=0$$

we see that the eigenvalues are determined by $2\sqrt{4\lambda-1}=n\pi$ for $n=1, 2, 3, \ldots$. Thus, the eigenvalues are $n^2\pi^2/4^2+1/4$ for $n=1, 2, 3, \ldots$, with corresponding eigenfunctions $e^{-x/2}\sin(n\pi x/2)$.

(b) The self-adjoint form is

$$\frac{d}{dx}[e^xy']+\lambda e^xy=0.$$

(c) An orthogonality relation is

$$\int_0^2 e^x\left(e^{-x/2}\sin\frac{m\pi}{2}x\right)\left(e^{-x/2}\cos\frac{n\pi}{2}x\right)dx=\int_0^2\sin\frac{m\pi}{2}x\,\cos\frac{n\pi}{2}x\,dx=0.$$

14. **(a)** An orthogonality relation is

$$\int_0^1\cos x_mx\cos x_nx\,dx=0$$

where $x_m\neq x_n$ are positive solutions of $\cot x=x$.

(b) Referring to Problem 1 we use a CAS to compute

$$\int_0^1(\cos 0.8603x)(\cos 3.4256x)\,dx=-1.8771\times10^{-6}\approx0.$$