Sec 12.4

2. Identifying $2p = 2$ or $p = 1$ we have

$$c_n = \frac{1}{2}\int_0^2 f(x)e^{-in\pi x}dx = \frac{1}{2}\int_1^2 e^{-in\pi x}dx = -\frac{1}{2in\pi}e^{-in\pi x}\Big|_1^2$$

$$= -\frac{1}{2in\pi}\left(e^{-2in\pi} - e^{-in\pi}\right) = -\frac{1}{2in\pi}[1-(-1)^n] = \frac{i}{2n\pi}[1-(-1)^n]$$

and

$$c_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}\,.$$

Thus

$$f(x) = \frac{1}{2} + \frac{i}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n}e^{in\pi x}.$$

4. Identifying $p = \pi$ we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx/\pi}dx = \frac{1}{2\pi}\int_0^{\pi} xe^{-inx/\pi}dx$$

$$= \frac{1}{2}\left(\frac{\pi}{n^2} + \frac{ix}{n}\right)e^{-inx/\pi}\Big|_0^{\pi} = \frac{\pi(1+in)}{2n^2}e^{-in} - \frac{\pi}{2n^2}$$

and

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}\,.$$

Thus

$$f(x) = \frac{\pi}{4} + \frac{\pi}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}\Big[(1+in)e^{-in} - 1\Big]e^{inx}.$$

6. Identifying $p = 1$ we have

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}\,dx = \frac{1}{2}\left[\int_{-1}^{0} e^{x}e^{-in\pi x}\,dx + \int_{0}^{1} e^{-x}e^{-in\pi x}\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{1-in\pi}e^{(1-in\pi)x}\Big|_{-1}^{0} - \frac{1}{1+in\pi}e^{-(1+in\pi)x}\Big|_{0}^{1}\right]$$

$$= \frac{e-(-1)^n}{e(1-in\pi)} + \frac{1-e^{-1}(-1)^n}{1+in\pi} = \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)}\,.$$

Thus

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2[e-(-1)^n]}{e(1+n^2\pi^2)}e^{in\pi x}.$$

11. **(a)** Adding $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$ we get $c_n + c_{-n} = a_n$. Subtracting, we get $c_n - c_{-n} = -ib_n$. Multiplying both sides by i we obtain $i(c_n - c_{-n}) = b_n$.

(b) From

$$a_n = c_n + c_{-n} = (-1)^n\frac{\sinh\pi}{\pi}\left[\frac{1-in}{n^2+1} + \frac{1+in}{n^2+1}\right] = \frac{2(-1)^n\sinh\pi}{\pi(n^2+1)}\,, \quad n = 0, 1, 2, \ldots$$

and

$$b_n = i(c_n - c_{-n}) = i(-1)^n\frac{\sinh\pi}{\pi}\left[\frac{1-in}{n^2+1} - \frac{1+in}{n^2+1}\right] = i(-1)^n\frac{\sinh\pi}{\pi}\left[-\frac{2in}{n^2+1}\right] = \frac{2(-1)^n n\sinh\pi}{\pi(n^2+1)}\,,$$

the Fourier series of f is

$$f(x) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2+1}\cos nx + \frac{n(-1)^n}{n^2+1}\sin nx\right].$$