Sec 12.2

4. $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_{0}^{1} x\,dx = \frac{1}{2}$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{0}^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{0}^{1} x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right]$$

6. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\,dx = \frac{5}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\,\Big|_0^{\pi} + \frac{2}{n}\int_{0}^{\pi} x\sin nx\,dx\right) = \frac{2}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\pi^2 - x^2\right)\sin nx\,dx$$

$$= \frac{\pi}{n}[(-1)^n - 1] + \frac{1}{\pi}\left(\frac{x^2 - \pi^2}{n}\cos nx\,\Big|_0^{\pi} - \frac{2}{n}\int_{0}^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}[1 - (-1)^n]$$

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^{n+1}\cos nx + \left(\frac{\pi}{n}(-1)^n + \frac{2[1 - (-1)^n]}{n^3\pi}\right)\sin nx\right]$$

16. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(e^x - 1)\,dx = \frac{1}{\pi}(e^\pi - \pi - 1)$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(e^x - 1)\cos nx\,dx = \frac{[e^\pi(-1)^n - 1]}{\pi(1+n^2)}$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(e^x - 1)\sin nx\,dx = \frac{1}{\pi}\left(\frac{ne^\pi(-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$

$$f(x) = \frac{e^\pi - \pi - 1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{e^\pi(-1)^n - 1}{\pi(1+n^2)}\cos nx + \left(\frac{n}{1+n^2}\left[e^\pi(-1)^{n+1} + 1\right] + \frac{(-1)^n - 1}{n}\right)\sin nx\right]$$

20. The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty}\frac{1 + (-1)^n}{\pi(1-n^2)}\cos\frac{n\pi}{2}$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3\cdot 5\pi} + \frac{2}{5\cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots.$$