Sec 12.1

6. $$\int_{\pi/4}^{5\pi/4} e^x \sin x\,dx = \left(\frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x\right)\Big|_{\pi/4}^{5\pi/4} = 0$$

12. For $m \neq n$, we use Problems 11 and 10:

$$\int_{-p}^{p} \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 2\int_{0}^{p} \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 0$$

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 2\int_{0}^{p} \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 0.$$

Also

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_{-p}^{p}\left(\sin\frac{(n-m)\pi}{p}x + \sin\frac{(n+m)\pi}{p}x\right)dx = 0,$$

$$\int_{-p}^{p} 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\,\Big|_{-p}^{p} = 0,$$

$$\int_{-p}^{p} 1\cdot\sin\frac{n\pi}{p}x\,dx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\,\Big|_{-p}^{p} = 0,$$

and

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x\,\cos\frac{n\pi}{p}x\,dx = \int_{-p}^{p}\frac{1}{2}\sin\frac{2n\pi}{p}x\,dx = -\frac{p}{4n\pi}\cos\frac{2n\pi}{p}x\,\Big|_{-p}^{p} = 0.$$

For $m = n$

$$\int_{-p}^{p} \cos^2\frac{n\pi}{p}x\,dx = \int_{-p}^{p}\left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

$$\int_{-p}^{p} \sin^2\frac{n\pi}{p}x\,dx = \int_{-p}^{p}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = p,$$

and

$$\int_{-p}^{p} 1^2 dx = 2p$$

so that

$$\|1\| = \sqrt{2p}\,, \quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}\,, \quad \text{and} \quad \left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}\,.$$

16. Using the facts that ϕ_0 and ϕ_1 are orthogonal to ϕ_n for $n > 1$, we have

$$\begin{aligned}\int_a^b (\alpha x + \beta)\phi_n(x)\,dx &= \alpha \int_a^b x\phi_n(x)\,dx + \beta \int_a^b 1 \cdot \phi_n(x)\,dx \\ &= \alpha \int_a^b \phi_1(x)\phi_n(x)\,dx + \beta \int_a^b \phi_0(x)\phi_n(x)\,dx \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0\end{aligned}$$

for $n = 2, 3, 4, \ldots$.

18. Setting

$$0 = \int_{-2}^{2} f_3(x) f_1(x)\,dx = \int_{-2}^{2} \left(x^2 + c_1 x^3 + c_2 x^4\right) dx = \frac{16}{3} + \frac{64}{5} c_2$$

and

$$0 = \int_{-2}^{2} f_3(x) f_2(x)\,dx = \int_{-2}^{2} \left(x^3 + c_1 x^4 + c_2 x^5\right) dx = \frac{64}{5} c_1$$

we obtain $c_1 = 0$ and $c_2 = -5/12$.