King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 301 Final Exam
Semester I, 2006- (061)
Dr. Faisal Fairag

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| Serial NO: | |
| ID: | KEY |
| Name: | KEY |

| Q | | Points |
|---|---|---|
| 1(a) | | 5 |
| 1(b) | | 10 |
| 1(c) | | 5 |
| 1(d) | | 10 |
| 2(a) | | 10 |
| 2(b) | | 6 |
| 3(a) | | 14 |
| 3(b) | | 14 |
| 3(c) | | 12 |
| 4(a) | | 8 |
| 4(b) | | 26 |
| 5 | | 16 |
| 6 | | 30 |
| 7 | | 35 |
| 8 | | 35 |
| 9 | | 14 |
| **Total** | | **250** |

Say a prayer & Good luck ☺

(1) (a) Find $L[f(t)]$ where $f(t) = 10\cosh(2t)$.

$\mathcal{L}\left[10\cosh(2t)\right] = 10\,\mathcal{L}\left[\cosh(2t)\right] = \dfrac{10\,s}{s^2-4}$

(b) Find $L[f(t)]$ where $f(t) = \begin{cases} -1 & 0 < t < 2 \\ 0 & t \geq 2 \end{cases}$.

$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^2 -e^{-st}\,dt$

$= -\int_0^2 e^{-st}\,dt = -\left[\dfrac{e^{-st}}{-s}\right]_0^2$

$= \left[\dfrac{e^{-st}}{s}\right]_0^2 = \dfrac{e^{-2s}}{s} - \dfrac{1}{s} = \dfrac{(e^{-2s}-1)}{s}$

(c) Find $L^{-1}\left[\dfrac{10}{s^2+25}\right]$. $= 2\,\mathcal{L}^{-1}\left[\dfrac{5}{s^2+25}\right]$ $\quad k = 5$

$= 2\sin(5t)$

(d) Find $L\left[e^{-t}\sin 3t\right]$. $= F(s-(-1)) = F(s+1)$

where $f(t) = \sin(3t)$, $a = -1$, $F(s) = \dfrac{3}{s^2+9}$

$= F(s+1) = \dfrac{3}{(s+1)^2+9}$

(2) (a) Compute $\nabla f(x,y), D_u f(x,y)$ for

$$f(x,y) = xy + 5y, \quad u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

$\frac{\partial f}{\partial x} = y$ 1 , $\frac{\partial f}{\partial y} = x + 5$ 1

$\nabla f(x,y) = (y)i + (x+5)j$ 2

$D_u f(x,y) = \nabla f \cdot u$

$= \left[yi + (x+5)j\right] \cdot \left[\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\right]$ 3

$= \frac{y}{\sqrt{2}} + \frac{x+5}{\sqrt{2}} = \frac{x+y+5}{\sqrt{2}}$ 3

(b) Compute *curl* F for

$$F(x,y,z) = (x^2 + y^2)i + (y^2 + z^2)j + (z^2 + x^2)k.$$

$\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & y^2+z^2 & z^2+x^2 \end{vmatrix} = (-2z)i + (-2x)j + (-2y)k$

$= -2zi - 2xj - 2yk$

2 2 2

(3) (a) Use Green's Theorem to evaluate the line integral

$\oint_c (x^2 y)dx + (\frac{1}{3}x^3 + x)dy$ where C is the unit circle.

$P(x,y) = x^2 y \quad , \quad Q(x,y) = \frac{1}{3}x^3 + x$

$\frac{\partial P}{\partial y} = x^2$ 3 $\quad , \quad \frac{\partial Q}{\partial x} = x^2 + 1$ 3

By Green's Theorem

$\oint = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R dA$ 3

$= \text{Area of the unit circle} = \pi (1)^2 = \pi$

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(b) Use Divergence Theorem to evaluate $\iint_S (F \cdot n)ds$ where

$F = xi + 2yj + 2z\ k$, D the region bounded by the sphere

$x^2 + y^2 + z^2 = 4$.

$\operatorname{div} F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z)$

$= 1 + 2 + 2 = 5$ 4

$x^2 + y^2 + z^2 = 4$ is a sphere of radius $2 \Rightarrow$ volume $= \frac{4}{3}\pi 2^3$

By Divergence Theorem 4

$\iint_S (F \cdot n)ds = \iiint_D \operatorname{div} F dV = \iiint_D 5\, dV = 5\,(\text{Volume of sphere})$

$= 5\left(\frac{4}{3}\pi 8\right) = \frac{160}{3}\pi$

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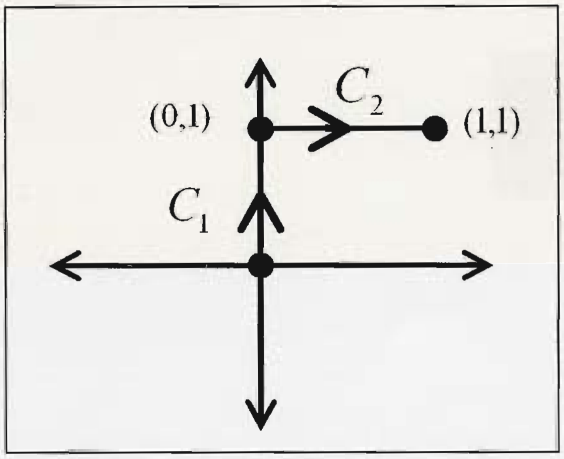
3(c) Evaluate $\oint_C (x^2 - y^2)dx + (x^2 + y^2)dy$ on the curve $C = C_1 \cup C_2$. (sec# 9.9)




$$\boxed{C_1: \quad x = 0 \quad dx = 0 \qquad 0 < t < 1 \\ \quad\; y = t \quad dy = dt}$$

$$\oint_{C_1} = \int_0^1 (0 - t^2)\cdot 0 + (0 + t^2)\,dt$$

$$= \int_0^1 t^2\,dt = \left[\tfrac{1}{3}t^3\right]_0^1 = 1/3 \quad 3$$

$$\boxed{C_2: \quad x = t \quad dx = dt \qquad 0 < t < 1 \\ \quad\; y = 1 \quad dy = 0}$$

$$\oint_{C_2} = \int_0^1 (t^2 - 1)\,dt + (t^2 + 1)\cdot 0$$

$$= \left[\tfrac{1}{3}t^3 - t\right]_0^1 = \tfrac{1}{3} - 1 = -2/3 \quad 3$$

Hence, $\oint_C = \oint_{C_1} + \oint_{C_2} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$ 6

(4) (a) Find a solution of the differential equation

$$(1-x^2)y'' - 2xy' + 3(3+1)y = 0 \qquad \text{———— (1)}$$

on the interval [-1,1].

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Equation (1) is a Legendre equation with $n=3$

Hence, $P_3(x)$ is a solution for (1) 4

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

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(b) Write out the first two nonzero terms in the Fourier-Legendre expansion of

$$f(x) = x^2 \qquad -1 < x < 1 \qquad C_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx$$

[ Hint: f(x) is an even function]

$P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2-1)$, $P_3 = \frac{1}{2}(5x^3-3x)$, $P_4 = \frac{1}{8}(35x^4-30x^2+3)$

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}(2)\int_0^1 x^2\,dx = \frac{1}{3}[x^3]_0^1 = 1/3 \quad 5$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} x^3\,dx = 0 \; ; \quad 5 \qquad c_2 = \frac{5}{2}\int_{-1}^{1} x^2 \cdot \frac{1}{2}(3x^2-1)\,dx$$

even

$$C_2 = \frac{5}{2}\int_{-1}^{1} x^2 \cdot \frac{1}{2}\cdot(3x^2-1)\,dx = \frac{5}{4}\int_{-1}^{1}(3x^4-x^2)\,dx = \frac{5}{4}(2)\int_0^1(3x^4-x^2)\,dx$$

$$= \frac{5}{2}\left[\frac{3x^5}{5} - \frac{1}{3}x^3\right]_0^1 = \frac{5}{2}\left[\frac{3}{5}-\frac{1}{3}\right] = \frac{5}{2}\left[\frac{9-5}{15}\right] = \frac{5}{2}\cdot\frac{4}{15} = \frac{2}{3} \quad 5$$

$$f(x) = C_0P_0(x) + c_2P_2(x) = \frac{1}{3} + \frac{2}{3}\left[\frac{1}{2}(3x^2-1)\right] \quad 11$$

(5) Find the complex Fourier series of $f$ on the interval (-2,2).

$$f(t)=\begin{cases}-1 & -2<x<0 \\ 1 & 0<x<2\end{cases}. \qquad p=2 \qquad \text{(sec\# 12.4)}$$

$$C_n=\frac{1}{4}\int_{-2}^{2} f(x)\, e^{\frac{-in\pi x}{2}}\,dx=\frac{1}{4}\left[\int_{-2}^{0}+\int_{0}^{2}\right]$$

$n=\pm1, \pm2, \pm3, \ldots$

$$\int_{-2}^{0}=\int_{-2}^{0}-e^{\frac{-in\pi x}{2}}\,dx=\left[-e^{\frac{-in\pi x}{2}}\cdot\frac{2}{-in\pi}\right]=\frac{2}{in\pi}\left[1-e^{in\pi}\right]$$

$$=\frac{2}{in\pi}\left[1-(\cos n\pi+i\sin n\pi)\right]=\frac{2}{in\pi}\left[1-(-1)^n\right]$$

$$=\frac{2i}{n\pi}\left[(-1)^n-1\right] \quad 3$$

$$\int_{0}^{2}=\int_{0}^{2}e^{\frac{-in\pi x}{2}}\,dx=\frac{2}{-in\pi}\left[e^{\frac{-in\pi x}{2}}\right]_0^2=\frac{2}{-in\pi}\left[e^{-in\pi}-1\right]$$

$$=\frac{2}{-in\pi}\left[(\cos n\pi-i\sin n\pi)-1\right]=\frac{2}{-in\pi}\left[(-1)^n-1\right]$$

$$=\frac{2i}{n\pi}\left[(-1)^n-1\right] \quad 3$$

Hence, $C_n=\frac{1}{4}\left[\frac{4i}{n\pi}\left((-1)^n-1\right)\right]=\frac{i}{n\pi}\left[(-1)^n-1\right]$ 3

$$C_0=\frac{1}{4}\int_{-2}^{2}f(x)\,dx=\frac{1}{4}\left[\int_{-2}^{0}-dx+\int_{0}^{2}dx\right]=\frac{1}{4}\left[-2+2\right]=0 \quad 3$$

Now, $f(x)=\sum_{n=-\infty}^{+\infty} C_n\, e^{\frac{in\pi x}{2}}$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{i}{n\pi}\left[(-1)^n-1\right]e^{\frac{in\pi x}{2}} \quad 4$$

(6) Use the Fourier Cosine transform to solve the boundary-value problem.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<\pi, \quad y>0 \qquad \text{——— (1)}$$

$$u(0,y)=h(y), \qquad \left.\frac{\partial u}{\partial x}\right|_{x=\pi}=0, \quad y>0 \qquad \text{——— (2)}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0, \quad 0<x<\pi \qquad \text{——— (3)}$$

where $$h(y)=\begin{cases}1 & 0<y<1\\ 0 & y>1\end{cases}.$$

We use the Fourier Cosine transform with respect to y to obtain

$$\mathcal{F}_c[u_{xx}]+\mathcal{F}_c[u_{yy}]=0$$

$$\frac{d^2U}{dx^2}-\alpha^2 U-u_y(x,0)=0 \qquad \text{——— (4)} \quad 4$$

Use (3) in equation (4): $\frac{d^2U}{dx^2}-\alpha^2 U=0$ ——— (5) 4

The solution of equation (5): $U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x$ — (6) 4

Take $\mathcal{F}_c$ of equations (2): $U(0,\alpha)=H(\alpha)$ ; $U'(\pi,\alpha)=0$ — (7)

where $U'=\frac{dU}{dx}$ and $H(\alpha)=\mathcal{F}_c(h(y))$

(7) $\Rightarrow$ $H(\alpha)=U(0,\alpha)=c_1$ and $0=c_1\sinh\alpha\pi+c_2\cosh\alpha\pi$ — (8)

Now, $c_1=H(\alpha)$ 4 , $c_2=-(H(\alpha)\sinh\alpha\pi)/(\cosh\alpha\pi)$ 4 — (9)

$$H(\alpha)=\int_0^\infty h(y)\cos\alpha y\,dy=\int_0^1\cos\alpha y\,dy=\left[\frac{\sin\alpha y}{\alpha}\right]_0^1=\frac{\sin\alpha}{\alpha} \quad \text{— (10)} \quad 4$$

Then, (6), (9), (10) $\Rightarrow$ $\boxed{U(x,\alpha)=\left(\frac{\sin\alpha}{\alpha}\right)\cosh\alpha x-\left(\frac{\sin\alpha}{\alpha}\right)\tanh\alpha\pi\cdot\sinh\alpha x}$ 4

Use the Inverse Fourier Cosine transform to find $u(x,y)$

$$\boxed{u(x,y)=\frac{2}{\pi}\int_0^\infty\left(\frac{\sin\alpha}{\alpha}\right)\left[\cosh\alpha x-\tanh\alpha\pi\sinh\alpha x\right]\cos\alpha y\,d\alpha} \quad 6$$

(7) Use the method of separation of variables to solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad 0<x<1 \quad 0<y<1 \quad \text{———} (1)$$

for a rectangle plate subject to the boundary conditions

$$u(0,y)=0, \quad u(1,y)=0 \quad \text{———} (2)$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0, \quad u(x,1)=h(x) \quad \text{———} (3)$$

where $h(x)=1$. Use $u = X(x)\cdot Y(y) \Rightarrow X''Y + XY''=0$

$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \Rightarrow X''+\lambda^2 X = 0$ and $Y''-\lambda^2 Y = 0$ —(4)

$(2) \Rightarrow X(0)Y(y)=0,\ X(1)\cdot Y(y)=0 \Rightarrow X(0)=0,\ X(1)=0$ —(5)

$(3) \Rightarrow X(x)\cdot Y'(0)=0 \Rightarrow Y'(0)=0$ —— (6)

3 3 3

Now, we have $X''+\lambda^2 X=0,\ X(0)=0,\ X(1)=0$ ——— (ode1)

$Y''-\lambda^2 Y=0,\ Y'(0)=0$ ——— (ode2)

3 3

The solution of (ode1) is; $X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$X(0)=0,\ X(1)=0 \Rightarrow c_1=0,\ \sin\lambda=0 \Rightarrow \lambda = n\pi$

$X_n(x) = \sin(n\pi x)$ 4 ——— (7)

The solution of (ode2) is : $Y = c_3\cosh\lambda y + c_4 \sinh\lambda y$

$Y' = c_3\lambda\sinh\lambda y + c_4\lambda\cosh\lambda y;\ Y'(0)=0 \Rightarrow c_4\lambda=0 \Rightarrow c_4=0$

$Y_n(y) = \cosh(n\pi y)$ 4 ——— (8)

From (7), (8) $\Rightarrow u_n(x,y) = \sin(n\pi x)\cdot\cosh(n\pi y)$

$$\boxed{u_n(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cdot\cosh(n\pi y)}$$ 4

$u(x,1) = h(x) \Rightarrow 1 = h(x) = \sum_{n=1}^{\infty} A_n\cosh(n\pi)\cdot\sin(n\pi x)$ —— (9)

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(9) is the Fourier Sine series of the function $h(x)=1$

$\Rightarrow A_n\cosh(n\pi) = 2\int_0^1 \sin(n\pi x)\,dx = 2\frac{1-(-1)^n}{n\pi} \Rightarrow \boxed{A_n = \frac{2[1-(-1)^n]}{n\pi\cosh(n\pi)}}$

(8) Find the solution $u(r,\theta)$ for the problem in Example 1 section 14.3 . If $f(\theta)=\cos\theta, \quad 0<\theta<\pi$.

[Hint: $p_1(\cos\theta)=\cos\theta. \quad p_1(x), p_2(x)$, are orthogonal.

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From page 731, The solution is

$$u(r,\theta)=\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \quad (*)$$

where $A_n = \frac{2n+1}{2c^n}\int_0^{\pi} f(\theta) P_n(\cos\theta)\sin\theta\, d\theta$

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi} \cos\theta \cdot P_n(\cos\theta)\sin\theta\, d\theta \quad (1) \quad 5$$

$x=\cos\theta \Rightarrow dx = -\sin\theta\, d\theta \quad (2)$

use (2) in (1) give: $A_n = \frac{2n+1}{2c^n}\int_1^{-1} x \cdot P_n(x) \cdot (-dx)$ 5

$$\Rightarrow A_n = \frac{2n+1}{2c^n}\int_{-1}^{1} x P_n(x)\, dx = \frac{2n+1}{2c^n}\int_{-1}^{1} P_1(x)P_n(x)\, dx$$

But $P_n(x)$ and $P_1(x)$ are orthogonal for $n\neq 1$ 4

$$\Rightarrow \int_{-1}^{1} P_1(x)P_n(x)\, dx = 0 \quad \text{for } n\neq 1$$

$$\Rightarrow A_n = 0 \quad \text{for } n\neq 1 \quad 7 \quad (3)$$

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Now, $A_1 = \frac{3}{2c}\int_{-1}^{1} P_1^2(x)\, dx = \frac{3}{2c}\int_{-1}^{1} x^2\, dx = \frac{3}{2c}\left[\frac{1}{3}+\frac{1}{3}\right] = \frac{1}{c}$ $\rightarrow (4)$

$(*), (3), (4) \Rightarrow u(r,\theta) = \frac{1}{c} r P_1(\cos\theta)$

$$\boxed{u(r,\theta) = \frac{1}{c} r \cos\theta} \quad 7$$

(9) Find the Fourier integral representation of the function *f*.

$$f(t) = \begin{cases} 0 & x < -2 \\ 1 & -2 < x < 2 \\ 0 & x > 2 \end{cases}$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] \, d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = \int_{-2}^{2} \cos \alpha x \, dx = \left[ \frac{\sin \alpha x}{\alpha} \right]_{-2}^{2}$$

$$= \frac{\sin 2\alpha - \sin(-2\alpha)}{\alpha} = \frac{2 \sin(2\alpha)}{\alpha}$$ 4

$$B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx = \int_{-2}^{2} \underbrace{\sin \alpha x}_{\text{odd}} \, dx = 0$$ 4

Hence,

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{2 \sin(2\alpha) \cos \alpha x}{\alpha} \, d\alpha}$$ 6