King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 301 Exam III
Semester I, 2006- (061)
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Serial NO:	**B**
ID:	
Name:	

Q	1	2	3	4	5	6	TOTAL
	(4 each) **39**	**22**	**15**	**15**	**20**	**22**	

Say a prayer & Good luck ☺

(1) **(In part (a) – (g)),**

Let $f(x)=\begin{cases}\cos x & 0<x<\frac{3\pi}{2}\\ \sin x & \frac{3\pi}{2}\le x<2\pi\end{cases}$, and let

$FS_1(x)$ = half-range cosine expansion of f(x).
$FS_2(x)$ = half-range sine expansion of f(x).
$FS_3(x)$ = half-range Fourier series expansion of f(x).
and
$\overline{FS}_1(x)$ = the periodic extension of $FS_1(x)$.
$\overline{FS}_2(x)$ = the periodic extension of $FS_2(x)$.
$\overline{FS}_3(x)$ = the periodic extension of $FS_3(x)$.

TRUE or FALSE:

(a)	$\overline{FS}_1(10\pi)=\overline{FS}_2(10\pi)$.	T	F
(b)	$\overline{FS}_2\left(\frac{19\pi}{2}\right)\neq\overline{FS}_3\left(\frac{27\pi}{2}\right)$.	T	F
(c)	$\overline{FS}_3(12\pi)\neq\overline{FS}_2(12\pi)$.	T	F
(d)	g(x) is a periodic function with period 4π where $g(x)=\overline{FS}_3(x)+\overline{FS}_2(x)$.	T	F
(e)	The graph of $FS_2(x)$ has spikes near the discontinuities at $x=\frac{3\pi}{2}$.	T	F
(f)	The coefficient of the term containing cosx in the series $FS_1(x)$ equals $\frac{2}{\pi}$.	T	F
(g)	The coefficient of the term containing cos(3x) in the series $FS_2(x)$ equals $\frac{3}{2\pi}$.	T	F

TRUE or FALSE:

(h)	The surface integral of the normal component of the curl of a conservative vector field **F** over a surface ***S*** is equal to zero.	T	F
(i)	For a two-dimensional vector field **F** in the plane z = 0, Stokes' theorem is the same as Green's theorem.	T	F

(2) The transverse displacement u(x,t) of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < L, \quad t > 0.$$

If the beam is **simply supported,** as shown in Figure 13.12, the boundary and initial condition are

$$u(0,t) = 0, \qquad u(L,t) = 0, \quad t > 0$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0, \quad t > 0$$

$$u(x,0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < \mathrm{x} < \mathrm{L}.$$

Solve for u(x,t). [*Hint: For convenience use* λ^4 *instead of* λ^2 *when separating variables.*]

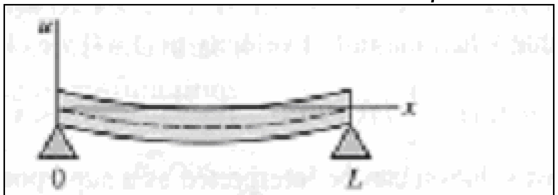


Figure 13.12 Simply supported beam

(3) Solve the heat equation subject to the condition: (Assume a rod of length 2π).

$u(0,t) = 0$, $u(2\pi,t) = 0$

$$u(x,0) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases} \qquad \text{Then find } \left.\frac{\partial u}{\partial x}\right|_{(t,x)=(0,0)}$$

(4) Use separation of variable to find product solution of $\frac{\partial^2 u}{\partial x \partial y} = u$.[*Hint: For convenience use* $-\lambda$]

(5) Expand $f(x) = 2x^2 - 1, \ -1 < x < 1$, in a Fourier series.

(6) If $F = \frac{1}{3}x^3 i + \frac{1}{3}y^3 j + \frac{1}{3}z^3 k$, use the divergence theorem to evaluate $\iint_S (F \cdot n) ds$ where S is the surface of the region bounded by $x^2 + y^2 = 1$, z = 0, z = 1.

#1)

(a) F	(b) T	(c) F	(d) T	(e) F	FORM A
(f) F	(g) F	(h) T	(i) T		
(a) T	(b) F	(c) T	(d) T	(e) F	FORM B
(f) F	(g) F	(h) T	(i) T		

#2) By separation variables: Assume that

$$u(x,t) = X(x)\cdot T(t)$$

So we have

$$a^2 X'''' \cdot T + X \cdot T'' = 0$$

which give

$$\frac{X''''}{X} = -\frac{T''}{a^2 T} = \lambda^4$$

and the two ode's are

$$X'''' - \lambda^4 X = 0 \text{ (eq*)} \quad \text{and} \quad T'' + a^2 T = 0 \text{ (eq**)}$$

and the solutions are

(1) — $X(x) = c_1 \cosh \lambda x + c_2 \sinh \lambda x + c_3 \cos \lambda x + c_4 \sin \lambda x$

(2) — $T(t) = c_5 \cos(\lambda^2 a t) + c_6 \sin(\lambda^2 a t)$

Now, the boundary conditions $u(0,t)=0$ and $u(4\,t)=0$ give

$$X(0) = 0 \text{ —(3)} \quad \text{and} \quad X(L) = 0 \text{ —(4)}$$

And, the boundary conditions $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0$, $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0$

$$X'(x) = \lambda c_1 \sinh \lambda x + \lambda c_2 \cosh \lambda x - c_3 \lambda \sin \lambda x + c_4 \lambda \cos \lambda x$$

$$X''(x) = c_1 \lambda^2 \cosh \lambda x + c_2 \lambda^2 \sinh \lambda x - c_3 \lambda^2 \cos \lambda x - c_4 \lambda^2 \sin \lambda x$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\cdot T(t) \Rightarrow X''(0) = 0 \text{ —(5)}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} \Rightarrow X''(L) = 0 \text{ —(6)}$$

Now, equations (1), (3), (5) $\Rightarrow$

$$c_1 + c_3 = 0 \quad \text{and} \quad c_1\lambda^2 - c_3\lambda^2 = 0$$

$$\Rightarrow c_1 = 0 \quad \text{and} \quad c_3 = 0.$$ 2

Hence,

$$X(x) = c_2 \sinh \lambda x + c_4 \sin \lambda x$$

$$X''(x) = c_2 \lambda^2 \sinh \lambda x - c_4 \lambda^2 \sin \lambda x$$

equations (4) and (6) $\Rightarrow$

$$c_2 \sinh \lambda L + c_4 \sin \lambda L = 0 \quad —(6)$$

$$c_2 \lambda^2 \sinh \lambda L - c_4 \lambda^2 \sin \lambda L = 0 \quad —(7)$$

(6) and (7) $\Rightarrow$ $c_4 \sin \lambda L = 0 \Rightarrow \sin \lambda L = 0$

$$\Rightarrow \lambda L = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots}$$ 3

$\lambda = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$ eigenvalues, $X = \sin \frac{n\pi}{L} x$ eigenfunctions $\longrightarrow (8)$

equations (eq **) and (8) give

3 $$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n^2\pi^2}{L^2} at + B_n \sin \frac{n^2\pi^2}{L^2} at \right] \sin \frac{n\pi}{L} x} \quad (9)$$

From (9),

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$ (half-range expansion in sine series)

we get 4 $$\boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx} \quad (10)$$

From (9),

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[-A_n \frac{n^2\pi^2 a}{L^2} \sin \frac{n^2\pi^2 at}{L^2} + B_n \frac{n^2\pi^2 a}{L^2} \cos \frac{n^2\pi^2}{L^2} at \right] \sin \frac{n\pi}{L}$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) = \sum_{n=1}^{\infty} B_n \frac{n^2\pi^2 a}{L^2} \sin \frac{n\pi}{L} x$$ (half-range expansion in cosine series)

we get $$B_n \frac{n^2\pi^2 a}{L^2} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

6 $$\boxed{B_n = \frac{2L}{n^2\pi^2 a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx} \quad (11)$$

#3) From page 698, The solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \, e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \cdot \sin \frac{n\pi}{L} x \quad \text{———} (1)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \quad \text{———} (2)$$

we have $L = 2\pi$ and $f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$

$$A_n = \frac{2}{2\pi} \int_0^{2\pi} f(x) \sin \frac{nx}{2} \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} + \cancel{\int_{\pi}^{2\pi}}^{\,\to 0} \right] =$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin \frac{nx}{2} \, dx = \frac{1}{\pi} \left\{ \left[-\cos \frac{nx}{2} \right] \frac{2}{n} \right\}_0^{\pi}$$

$$= -\frac{2}{n\pi} \left[\cos \frac{n\pi}{2} - 1 \right] = \frac{2}{n\pi} \left[1 - \cos \frac{n\pi}{2} \right]$$ 5

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2\left(1 - \cos \frac{n\pi}{2}\right)}{n\pi} \, e^{-k\left(\frac{n^2\pi^2}{4\pi^2}\right)t} \cdot \sin \frac{n\pi}{2\pi} x$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2\left[1 - \cos \frac{n\pi}{2}\right]}{n\pi} \, e^{-k\left(\frac{n^2}{4}\right)t} \cdot \sin \frac{n}{2} x}$$ 5

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{2\left(1 - \cos \frac{n\pi}{2}\right)}{n\pi} \, e^{-k\left(\frac{n^2}{4}\right)t} \cdot \cos \frac{n}{2} x \cdot \frac{n}{2}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(t,x)=(0,0)} = \sum_{n=1}^{\infty} \frac{\left(1 - \cos \frac{n\pi}{2}\right)}{\pi}$$ 5

#4) Assume $u(x,y) = X(x) \cdot Y(y)$

$$\frac{\partial u}{\partial y} = X Y' \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = X' Y'$$

from PDE: $X' Y' = X Y \Rightarrow \frac{X'}{X} = \frac{Y}{Y'}$

$$\boxed{\frac{X'}{X} = \frac{Y}{Y'} = -\lambda}$$

If $\lambda = 0 \Rightarrow \left\{ \begin{array}{l} X' = 0 \Rightarrow X(x) = c_1 \\ Y = 0 \Rightarrow Y(y) = 0 \end{array} \right\} \Rightarrow u = 0$ 3

If $\lambda \neq 0 \Rightarrow \left\{ \begin{array}{l} X' + \lambda X = 0 \\ Y' + (\frac{1}{\lambda}) Y = 0 \end{array} \right\} \Rightarrow \begin{array}{l} X(x) = c_1 e^{-\lambda x} \\ Y(y) = c_2 e^{-\frac{1}{\lambda} y} \end{array}$ 3 3

$$u(x,y) = c_1 c_2 e^{-\lambda x} \cdot e^{-\frac{1}{\lambda} y}$$

$$\boxed{u(x,y) = A e^{-(\lambda x + \frac{1}{\lambda} y)}}$$ 6

#5) $f(x) = 2x^2 - 1 \qquad -1 < x < 1 \quad , \; P = 1$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \int_{-1}^{1} (2x^2-1)\,dx = -\tfrac{2}{3}$$ 3

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = \int_{-1}^{1} (2x^2-1)\cos(n\pi x)\,dx$$

$$= \left[\frac{2x^2-1}{n\pi}\sin(n\pi x) + \frac{4x}{n^2\pi^2}\cos(n\pi x) - \frac{4}{n^3\pi^3}\sin n\pi x\right]_{-1}^{1}$$

$2x^2-1$		$\cos(n\pi x)$
$4x$	$+$	$\frac{1}{n\pi}\sin(n\pi x)$
4	$-$	$\frac{-1}{n^2\pi^2}\cos(n\pi x)$
0	$+$	$\frac{-1}{n^3\pi^3}\sin(n\pi x)$

$$= \left[\frac{4}{n^2\pi^2}(-1)^n\right] - \left[\frac{-4}{n^2\pi^2}(-1)^n\right]$$

$$= \frac{8}{n^2\pi^2}(-1)^n$$ 5

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx = \int_{-1}^{1} (2x^2-1)\sin(n\pi x)\,dx = 0$$ 5

(Note: $f(x)$ even $\Rightarrow (2x^2-1)\sin(n\pi x)$ is odd)

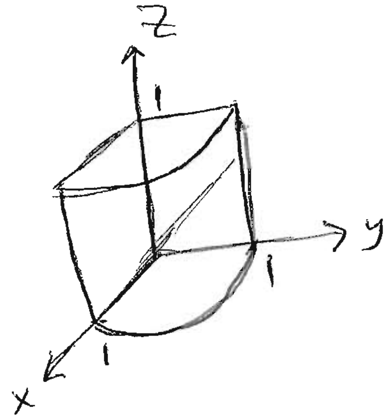
Hence,

$$\boxed{f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}(-1)^n \cos(n\pi x)}$$ 7

#6) $F = \frac{1}{3}x^3 i + \frac{1}{3}y^3 j + \frac{1}{3}z^3 k$

$$\operatorname{div} F = \frac{\partial}{\partial x}\left(\frac{1}{3}x^3\right) + \frac{\partial}{\partial y}\left(\frac{1}{3}y^3\right) + \frac{\partial}{\partial z}\left(\frac{1}{3}z^3\right)$$

$$= x^2 + y^2 + z^2$$ 2



$$\iint_{S} F \cdot n\, ds = \iiint_D \operatorname{div} F \, dV$$

$$= \iiint_D (x^2 + y^2 + z^2)\, dV$$ 4

Using cylindrical coordinates 8

$$= \int_0^{2\pi} \int_0^1 \int_0^1 (r^2 + z^2)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[r^3 z + \frac{1}{3} r z^3\right]_0^1 dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[r^3 + \frac{1}{3} r\right] dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{4} r^4 + \frac{1}{6} r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{4} + \frac{1}{6}\right] d\theta$$

$$= \int_0^{2\pi} \frac{5}{12}\, d\theta =$$

$$= \left[\frac{5}{12}\theta\right]_0^{2\pi} = \frac{5}{6}\theta$$ 10