King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 301 Exam II
Semester I, 2006- (061)
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Serial NO:	KEY
ID:	
Name:	KEY

FORM **A**

Q		Points
1		8
2		8
3		8
4		8
5		8
6		8
7		8
8		10
9		16
10		16
11		16
12		16
Total		130

Say a prayer & Good luck ☺

1) If $f(t)=e^{2t}(t-1)^2$ and $F(s)=L[f(t)]$, then F(1)= $F(5)$

$F(s)=\mathcal{L}[e^{2t}t^2 - e^{2t}2t + e^{2t}] = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}$

(a) $\frac{18}{7}$ (b) 0 (c) $\frac{139}{108}$ (d) $\frac{-7}{18}$ (e) none

A, B

$F(1) = \frac{2}{-1} - \frac{2}{1} + \frac{1}{-1} = -2-2-1 = -5$

$F(5) = \frac{2}{27} - \frac{2}{9} + \frac{1}{3} = \frac{2-6+9}{27} = \frac{5}{27}$

2) If $f(t)=L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$, then f(0)=

(a) $\frac{1}{2}$ (b) 0 (c) $\frac{1}{5}$ (d) $\frac{1}{\pi^2+5}$ (e) none

B, A

$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} = \frac{1}{s^2+4}\Big|_{s\to s+1} \Rightarrow f(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\Big|_{s\to s+1}\right] = \frac{1}{2}e^{-t}\sin 2t$

$f(0) = \frac{1}{2}(1)(0) = 0$

$f(\pi) = \frac{1}{2}e^{-\pi}\sin(2\pi) = 0$

✓ **3)** If $f(t)=L^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$, then f(3)=

(a) e^2-3 (b) 0 (c) e-2 (d) 1 (e) none

B A

$\frac{1}{s^2(s-1)} = \frac{-1}{s} + \frac{-1}{s^2} + \frac{1}{s-1} \Rightarrow f(t) = \mathcal{L}^{-1}\left\{-\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s-1}\right\}$

$f(t) = -\mathcal{U}(t-2) - (t-2)\mathcal{U}(t-2) + e^{t-2}\mathcal{U}(t-2)$

$= \mathcal{U}(t-2)\left[1-t+e^{t-2}\right]$

$f(3) = 1\cdot[1-3+e^1] = e-2$, $f(4) = (1)[-3+e^2] = e^2-3$

4) If $F(s) = L[te^{-3t}\cos 3t]$, then F(-3)=

(a) $\frac{-1}{9}$ (b) 0 (c) $\frac{1}{9}$ (d) e^{-9} (e) none

A B

$$F(s) = -\frac{d}{ds}\mathcal{L}[e^{-3t}\cos 3t] = -\frac{d}{ds}\left[\mathcal{L}[\cos 3t]\Big|_{s\to s+3}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+9}\Big|_{s\to s+3}\right] = -\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+9}\right] = -\frac{(s+3)^2+9-2(s+3)^2}{[(s+3)^2+9]^2}$$

$$= \frac{(s+3)^2-9}{[(s+3)^2+9]^2} \Rightarrow F(-3) = \frac{0-9}{[0+9]^2} = -\frac{1}{9}, \quad F(0) = \frac{9-9}{(9+9)^2} = 0$$

5) If $y(t)$ is the solution of the initial value problem : $y' + y = \delta(t-1)$, y(0)=2
Then $y(1) =$

$$sY(s) - y(0) + Y(s) = e^{-s}$$

$$Y(s)[s+1] = 2 + e^{-s} \Longrightarrow Y(s) = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$

(a) 2 (b) $\frac{2+e}{2}$ (c) 0 (d) 1 (e) none

B A

$$y(t) = 2e^{-t} + e^{-(t-1)}u(t-1)$$

$$y(1) = 2e^{-1} + 1 = \frac{2}{e} + 1 = \frac{2+e}{e}$$

$$y(2) = 2$$

6) If $h(t) = t * \cos t$, then $h(\pi) =$

$$h(\pi) = 1 - \cos\pi = 1-(-1) = 2$$

$$h(\tfrac{\pi}{2}) = 1 - \cos\tfrac{\pi}{2} = 1 - 0 = 1$$

(a) 2 (b) 1 (c) 0 (d) -1 (e) none

A B

$$h(t) = t * \cos t = \cos t * t = \int_0^t \cos\tau \cdot (t-\tau)\,d\tau$$

$$= \int_0^t t\cos\tau - \int_0^t \tau\cos\tau\,d\tau$$

$$= t[\sin\tau]_0^t - [\tau\sin\tau + \cos\tau]_0^t$$

$$= t\sin t - [t\sin t + \cos t - 1] = 1 - \cos t$$

τ	$\cos\tau$
1	$\sin\tau$
0	$-\cos\tau$

$\frac{1}{s^2}\cosh s = \left(\frac{e^{s} + e^{-s}}{2}\right)\frac{1}{s^2}$

$= \frac{1}{2}\left[\frac{1}{s^2}e^{s} + \frac{1}{s^2}e^{-s}\right]$

7) If $f(t) = L^{-1}\left\{\frac{1}{s^2}\cosh s\right\}$, then $f(2) =$

$f(t) = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-s}\right]$

(a) 10   (b) 5   (c) 1   (d) 0   (e) none

B   A

$= \frac{1}{2}(t+1)^2\mathcal{U}(t+1) + \frac{1}{2}(t-1)^2\mathcal{U}(t-1)$

$f(2) = \frac{1}{2}(3)^2(1) + \frac{1}{2}(1)^2(1) = \frac{9}{2} + \frac{1}{2} = 5$

$f(3) = \frac{1}{2}(4)^2(1) + \frac{1}{2}(4) = \frac{16}{2} + \frac{4}{2} = 8+2 = 10$

8) Evaluate the given Laplace transform without evaluating the integral $L\left\{t\int_0^t \tau e^{-\tau}d\tau\right\}$

Let $g(t) = \int_0^t \tau e^{-\tau}d\tau$

by Theorem 4.8

$\mathcal{L}\{t\,g(t)\} = -\frac{d}{ds}G(s)$ 2 ——— (1)

$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t \tau e^{-\tau}d\tau\right\}$

$= \frac{F(s)}{s}$ 2 by eq (7) pp 218

where $f(t) = te^{-t} \Rightarrow F(s) = -\frac{d}{ds}\mathcal{L}\{e^{-t}\}$

$= -\frac{d}{ds}\frac{1}{s+1} = \frac{1}{(s+1)^2}$

Now, $G(s) = \frac{F(s)}{s} = \frac{1}{s(s+1)^2}$ 2 ——— (2)

(1) and (2) give

$\mathcal{L}\{tg(t)\} = -\frac{d}{ds}\left[\frac{1}{s(s+1)^2}\right] = -\left[-\frac{1}{s^2}\frac{1}{(s+1)^2} - 2\frac{1}{(s+1)^3}\frac{1}{s}\right]$

$= \frac{2s^2+1}{s^2(s+1)^3}$ 4

9) Use the Laplace transform to solve the given initial value problem

$y'' + 4y = \sin t\, U(t-2\pi), \quad y(0) = 1, \quad y'(0) = 0$

Take $\mathcal{L}$

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\sin t\, \mathcal{U}(t-2\pi)]$$

$$[s^2 Y(s) - s y(0) - y'(0)] + 4Y(s) = e^{-2\pi s}\frac{1}{s^2+1}$$

$$s^2 Y(s) - s + 4Y(s) = \frac{e^{-2\pi s}}{s^2+1}$$

$$\boxed{Y(s) = \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}} \quad 2$$

By partial fraction $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{1/3}{s^2+1} - \dfrac{1/6}{s^2+4}$ 2

$$Y(s) = \frac{1}{3}\underbrace{e^{-2\pi s}\frac{1}{s^2+1}}_{1} - \frac{1}{6}\underbrace{e^{-2\pi s}\frac{1}{s^2+4}}_{2} + \underbrace{\frac{s}{s^2+4}}_{3} \quad \text{———} (*)$$

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+1}\right\} = \mathcal{U}(t-2\pi) f_1(t-2\pi) \quad \text{when } F_1(s) = \frac{1}{s^2+1}$$

$$= \mathcal{U}(t-2\pi)\sin(t-2\pi) \quad \text{———} 1 \quad 3$$

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+4}\right\} = \mathcal{U}(t-2\pi) f_2(t-2\pi) \quad \text{when } F_2(s) = \frac{1}{s^2+4}$$

$$= \mathcal{U}(t-2\pi)\sin 2(t-2\pi) \quad \text{———} 2 \quad 3$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t \quad \text{———} 3 \quad 2$$

$(*)$, 1, 2 and 3 give

$$y(t) = \cos 2t + \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]\mathcal{U}(t-2\pi)$$

3

10) Use the Laplace transform to solve

$$y'(t) = \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau, \quad y(0) = 1$$

Taking the $\mathcal{L}$ of the equation

$$s\mathcal{L}[y] - y(0) = \mathcal{L}[\cos t] + \mathcal{L}\left\{\int_0^t y(\tau)\cos(t-\tau)\,d\tau\right\}$$

$$sY(s) - 1 = \frac{s}{s^2+1} + \mathcal{L}\{y(t) * g(t)\}$$

where $g(t) = \cos t \Rightarrow G(s) = \frac{s}{s^2+1}$

$$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\cdot G(s)$$ 4

$$sY(s) - 1 = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$Y(s)\left[s - \frac{s}{s^2+1}\right] = 1 + \frac{s}{s^2+1}$$

$$Y(s)\left[\frac{s^3}{s^2+1}\right] = 1 + \frac{s}{s^2+1} \Rightarrow Y(s) = \frac{s^2+1}{s^3} + \frac{1}{s^2}$$

$$\boxed{Y(s) = \frac{1}{s} + \frac{1}{s^3} + \frac{1}{s^2}}$$ 4

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$\boxed{y(t) = 1 + \frac{1}{2}t^2 + t}$$ 8

$$\boxed{\textbf{11)}\ \text{If } f(t) = \mathcal{L}^{-1}\left[\frac{64s}{(s^2+4)^2}\right],\ \text{then find } f(0)}$$

Solution (1) From Example 4 in page 217,

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+k^2)^2}\right] = (\sin kt - kt\cos kt)/2k^3$$

with $k=2 \Rightarrow \mathcal{L}[\underbrace{\sin 2t - 2t\cos 2t}_{f(t)}] = \dfrac{16}{(s^2+4)^2}$ ——— (1)

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}\left\{\mathcal{L}[f(t)]\right\} = -\frac{d}{ds}\left[\frac{16}{(s^2+4)^2}\right]$$

$$= -\left[(16)(-2)(s^2+4)^{-3}(2s)\right] = \frac{64s}{(s^2+4)^3}$$

So, $\mathcal{L}[t\sin 2t - 2t^2\cos 2t] = \dfrac{64s}{(s^2+4)^3}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{64s}{(s^2+4)^3}\right] = t\sin 2t - 2t^2\cos 2t \Rightarrow f(0) = 0$$

Solution (2) We will use the result $\mathcal{L}^{-1}[F(s)G(s)] = f*g$

Let $F(s) = \dfrac{16}{(s^2+4)^2}$ and $G(s) = \dfrac{4s}{(s^2+4)} \Rightarrow g(t) = 4\cos 2t$

and $f(t) = \sin 2t - 2t\cos 2t$ (From example 4/pp217 or (1))

Hence, $\mathcal{L}^{-1}\left[\dfrac{64s}{(s^2+4)^3}\right] = f*g = \displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$

$$= 4\int_0^t [\sin 2\tau - 2\tau\cos 2\tau]\cos 2(t-\tau)\,d\tau$$

$$= 4\Big[\underbrace{\int_0^t \sin 2\tau\cos 2(t-\tau)\,d\tau}_{I_1} - \underbrace{\int_0^t 2\tau\cos 2\tau\cos 2(t-\tau)\,d\tau}_{I_2}\Big]$$

* use $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ to write I_1 as

$$I_1 = \frac{1}{2}\int_0^t [\sin 2t + \sin(4\tau - 2t)]\,dt = \frac{1}{2}t\sin 2t + 0 = \frac{1}{2}t\sin 2t$$

* use $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$ to write I_2 as

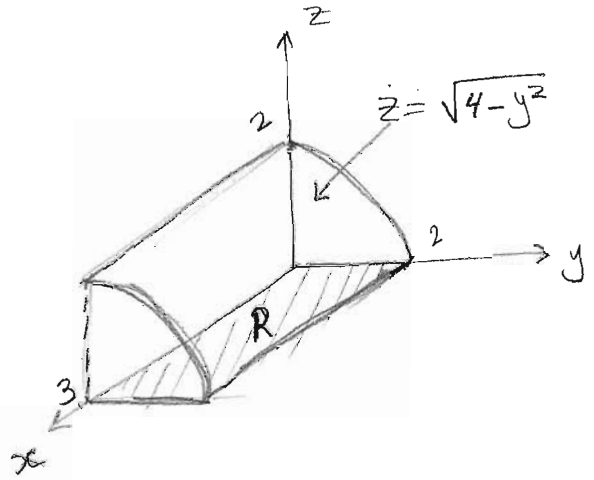
$$I_2 = -\int_0^t \tau[\cos 2t + \cos(4\tau - 2t)]\,d\tau$$

$$= -\int_0^t \tau\cos 2t\,d\tau - \int_0^t \tau\cos(4\tau-2t)\,d\tau = \left[-\frac{\tau^2}{2}\cos 2t\right]_0^t - \left[\frac{\tau\sin(4\tau-2t)}{4} + \frac{\cos(4\tau-2}{16}\right]^t$$

$$= -\frac{t^2}{2}\cos 2t - \frac{1}{4}t\sin 2t \Rightarrow f(t) = 4[I_1 + I_2] = t\sin 2t - 2t^2\cos 2$$

$$\Rightarrow f(0) = 0$$

12) Let $F(x,y,z) = xi + 2zj + yk$ represent the flow of a liquid. Find the flux of F through the surface S given by that portion of the cylinder $y^2 + z^2 = 4$ in the first octant bounded by x=0, x=3, y=0, z=0 oriented upward.



$$\text{Flux} = \iint_S F \cdot n \, dS$$

Step 1: Finding n

The surface is $g(x,y,z) = y^2 + z^2 - 4$

$$\nabla g = (2y)j + (2z)k$$

$$\|\nabla g\| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2+z^2}$$

$$\boxed{n = \frac{\nabla g}{\|\nabla g\|} = \frac{y}{\sqrt{y^2+z^2}} j + \frac{z}{\sqrt{y^2+z^2}} k}$$ 4

step2: $dS = \sqrt{1 + f_x^2 + f_y^2}\, dA$

The surface is $z = \sqrt{4-y^2} = f(x,y)$ (R is the projection on xy-p

$$f_x = 0 \quad , \quad f_y = \frac{-y}{\sqrt{4-y^2}}$$

$$dS = \sqrt{1 + \frac{y^2}{4-y^2}}\, dA = \sqrt{\frac{4}{4-y^2}} = \frac{2}{\sqrt{4-y^2}} dA$$

$$\boxed{dS = \frac{2}{\sqrt{4-y^2}} dA}$$ 5

Now, $$F \cdot n = \frac{2zy}{\sqrt{y^2+z^2}} + \frac{yz}{\sqrt{y^2+z^2}} = \frac{3yz}{\sqrt{y^2+z^2}}$$

$$\text{Flux} = \iint_S F \cdot n \, dS = \iint_R \frac{3y\sqrt{4-y^2}}{\sqrt{y^2+(4-y^2)}} \frac{2}{\sqrt{4-y^2}} dA$$

$$= \iint_R 3y \, dA = \int_0^3\int_0^2 3y \, dy\, dx = \int_0^3 \frac{3}{2} y^2 \Big|_0^2 dx = \int_0^3 6\, dx = 18$$ 7

The End ☺