(1) Find the solution of the linear system in **vector form**.

$$x + 2y + z = 0$$
$$\frac{1}{2}x + y + \frac{1}{2}z = 0$$
$$\frac{1}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 1/3 & 2/3 & 1/3 & 0 \end{array}\right] \xrightarrow[-1/3 R_1 + R_3]{-1/2 R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $y = w$, $z = t$

$\Rightarrow x = -2w - t$

$$A = \begin{bmatrix} -2w - t \\ w \\ t \end{bmatrix} = w\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(2) Write u as a linear combination of v1, v2 and v3:

u=[3;3;1;0], v1= [1;1;0;0], v2=[-1;0;1;1], v3=[1;3;3;2]

$$u = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_4]{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{c_3 = 1}$$

$$\boxed{c_2 = -2c_3 = -2}$$

$$\boxed{c_1 = 3 + c_2 + (-c_3) = 3 - 2 - 1 = 0}$$

$$u = 0\, v_1 + (-2\, v_2) + v_3$$

$$= -2\, v_2 + v_3$$

$$\underset{u}{\begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \end{bmatrix}} = \underset{c_1}{(0)} \underset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + \underset{c_2}{(-2)} \underset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} + \underset{c_3}{(1)} \underset{v_3}{\begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}}$$

(4) Write the matrix A as a product of elementary matrices.

A=[2,1;3,2]

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow[E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}]{-R_1 + R_2} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow[E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[E_3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}]{-2R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -1 & 3 & -2 \end{array}\right] \xrightarrow[E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}]{-R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 2 \end{array}\right] \xrightarrow[E_5 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}]{-R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right] = [I \mid A^{-1}]$$

$$A^{-1} = E_5 E_4 E_3 E_2 E_1$$

$$A = E^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(5) Determine whether the vectors v1, v2, v3 are linearly independent or dependent. If they are linearly independent , show this. otherwise find a nontrivial linear combination of them that is equal to the zero vector.

v1=[1;2;3;4] v2=[0;1;0;0] v3=[1;2;4;5]

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 3 & 0 & 4 & 0 \\ 4 & 0 & 5 & 0 \end{array}\right] \xrightarrow[-4R_1+R_4]{\substack{-2R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_3+R_4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

The only solution for this system is $c_1 = c_2 = c_3 = 0$

$\Rightarrow$ They are linearly independent

(3) Find the value of k so that the matrix A is row equivalent to the identity matrix I.

A=[1,0,1;0,1,k;2,1,1]

By theorem 7,

$$\boxed{\text{A is row equivalent to I}} \iff \boxed{\det(A) \neq 0}$$

$$\det(A) = |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & k \\ 2 & 1 & 1 \end{vmatrix} \overset{-C_1+C_2}{=\!=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 2 & 1 & -1 \end{vmatrix}$$

$$\underset{\text{rows}}{\overset{\text{expand along } R_1}{=\!=}} (1) \begin{vmatrix} 1 & k \\ 1 & -1 \end{vmatrix} = -1 - k \neq 0$$

$$\det(A) \neq 0 \iff \boxed{k \neq -1}$$

(6) (True or False)
Let A be an nxn invertible matrix and let E1 and E2 be an nxn elementary matrices. If B = E1 * inv(A) * E2 * A^2 then the matrix A is row equivalent to the matrix B. (. . . . . . .)
(Justify your answer)

Thm: $\boxed{\text{A is row equivalent to I}} \Longleftrightarrow \boxed{\det(A) \neq 0}$

E_1, E_2 elem. matrices $\Rightarrow$ E_1 and E_2 invertable

$\Leftrightarrow$ $|E_1| \neq 0, \ |E_2| \neq 0$ —(1)

A invertible $\Rightarrow$ $|A| \neq 0$ —— (2)

Now, $|B| = |E_1 A^{-1} E_2 A^2|$

$= |E_1| \cdot |A^{-1}| \cdot |E_2| \cdot |A^2|$

$= |E_1| \cdot \frac{1}{|A|} \cdot |E_2| \cdot |A|^2$

$= |E_1| \cdot |E_2| \cdot |A|$ —— (3)

(1), (2), (3) $\Rightarrow$ $|B| \neq 0$

$\Rightarrow$ B is row equivalent to I —(4)

$|A| \neq 0 \Rightarrow$ A is row equivalent to I —(5)

(4) and (5) $\Rightarrow$ A is row equivalent to B

(7) Let D be a 12x12 matrix.

$$D=\left[\begin{array}{c|c|c} A & I & 0 \\ \hline -I & B & -I \\ \hline A+C & I & C \end{array}\right] \quad \text{where} \quad A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad I=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{then find } \det(D)$$

$$|D| \underset{\substack{-R_1+R_9 \\ -R_2+R_{10} \\ -R_3+R_{11} \\ -R_4+R_{12}}}{=} \begin{vmatrix} A & I & O \\ -I & B & -I \\ C & O & C \end{vmatrix} \underset{\substack{-C_9+C_1 \\ -C_{10}+C_2 \\ -C_{11}+C_3 \\ -C_{12}+C_4}}{=} \begin{vmatrix} A & I & O \\ O & B & -I \\ O & O & C \end{vmatrix}$$

$$= |A|\cdot|B|\cdot|C| = (5)(1)(-1) = -5 \Rightarrow \boxed{|D| = -5}$$

$$|B| \underset{R_1+R_2}{=} \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 1$$

$$|C| \underset{-R_1+R_2}{=} \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

$$|A| \underset{-2R_2+R_1}{=} \begin{vmatrix} 0 & -3 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} -3 & -2 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} \underset{3R_2+R_1}{=}$$

$$= (-1)\begin{vmatrix} 0 & 4 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = (8-3) = 5$$

(8) True or False.

[1]	Any set of more than n vectors in R^n is linearly dependent	(T) (F)
[2]	The vectors v1, v2, … , vk are linearly dependent if and only if one of them is a linear combination of the others	(T) (F)
[3]	Any subset of a linearly independent set S = {v1, v2, … , vk} is a linearly independent set of vectors.	(T) (F)
[4]	Any set of less than n vectors in R^n is linearly independent	(T) (F)
[5]	Let v1 and v2 be vectors in R^5 and let v3=2*v2-v1, v4=2*v1-v2. Let W1=span{v1,v2} and W2=span{v3, v4} then W1 = W2	(T) (F)
[6]	In R^4, Span{e1, e2} = Span{v1, v2} where e1 and e2 are basic unit vectors and v1=[1;1;0;0], v2=[-1;1;0;0].	(T) (F)
[7]	If E is an elementary matrix of size 5x5 then the columns of the matrix E are linearly independent vectors in R^5.	(T) (F)
[8]	If E is an elementary matrix then E^T is also an elementary matrix.	(T) (F)

(4) In $\mathbb{R}^3$, $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. $\{v_1, v_2\}$ lin. dependent

(5) $v_1 = \frac{1}{3} v_3 + \frac{2}{3} v_4$ and $v_2 = \frac{2}{3} v_3 + \frac{1}{3} v_4$. v_1, v_2 are lin. combi of v_3, v_4
v_3, v_4 〃 〃 〃 〃 v_1, v_2

$\Rightarrow W_1 \subseteq W_2$, $W_2 \subseteq W_1$ $\Rightarrow W_1 = W_2$

(6) $e_1 = \frac{1}{2} v_1 - \frac{1}{2} v_2$ and $e_2 = \frac{1}{2} v_1 + \frac{1}{2} v_2$ $\Rightarrow$ Span(e_1, e_2) = Span(v_1, v_2)
$v_1 = e_1 + e_2$ and $v_2 = -e_1 + e_2$

(7) E elem $\Rightarrow$ E invertable $\Rightarrow$ Columns of E are lin. indep