5) $y' + 3x^2 y = 10x^2$

I.F. $= e^{\int 3x^2 dx} = e^{x^3}$

Multiply throughout by e^{x^3}

$e^{x^3} y' + 3x^2 e^{x^3} y = 10x^2 e^{x^3}$

$\Rightarrow \frac{d}{dx}(y e^{x^3}) = 10x^2 e^{x^3}$

Integrating both sides

$y e^{x^3} = 10 \int x^2 e^{x^3} dx$

$= 10\left[\frac{1}{3}\int 3x^2 e^{x^3} dx\right] = \frac{10}{3} e^{x^3} + C$

Hence $y e^{x^3} = \frac{10}{3} e^{x^3} + C$

or $y = \frac{10}{3} + C e^{-x^3}$.

13) $x^2 y' + x(x+2)y = e^x$

Re-write as $y' + \frac{x^2+2x}{x^2} y = \frac{e^x}{x^2}$

I.F $= e^{\int \frac{x^2+2x}{x^2} dx} = e^{\int (1+\frac{2}{x})dx} = e^{x+2\ln x}$

$= x^2 e^x$

Multiply the DE by I F

$y' x^2 e^x + (x^2+2x) e^x y = e^{2x}$

or $\frac{d}{dx}(x^2 e^x y) = e^{2x}$

Integrate both sides

$x^2 e^x y = \frac{1}{2} e^{2x} + C$

y can be found.

16) $y\,dx = (y e^y - 2x)\,dy$

[D.E is not linear in y].

$y \frac{dx}{dy} + 2x = y e^y$

or $\frac{dx}{dy} + \frac{2}{y} x = y e^y$

Linear D.E in x

I.F. $= e^{\int \frac{2}{y} dy} = e^{2\ln y} = y^2$

Multiplying by I.F and integrating

$y^2 x = \int y^3 e^y dy =$

$y^2 e^y - 3\int y^2 e^y dy = y^3 e^y - 3y^2 e^y + 6\int y e^y dy$

so $y^2 = y^3 e^y - 3y^2 e^y + 6y e^y - 6e^y + C$

or $x y^2 = e^y[y^3 - 3y^2 + 6y - 6] + C$

18) $\cos^2 x \sin x \frac{dy}{dx} + \cos^3 x\, y = 1$

Re-write as

$\frac{dy}{dx} + \frac{\cos^3 x}{\cos^2 x \sin x} y = \frac{1}{\cos^2 x \sin x}$

or $\frac{dy}{dx} + \cot x\, y = \frac{1}{\cos^2 x \sin x}$

I.F. $= e^{\int \cot x\, dx} = e^{\ln \sin x} = \sin x$

Multiply by I.F

$\sin x \frac{dy}{dx} + \cos x\, y = \frac{\cancel{\sin x}}{\cos^2 x \cancel{\sin x}}$

or $\frac{d}{dx}[y \sin x] = \sec^2 x$

Integrating

$y \sin x = \tan x + C$.

37) $y' - 2xy = 1$, $y(1) = 1$

I.F. $= e^{-2\int x dx} = e^{-x^2}$

Multiply by I.F and integrate

$y e^{-x^2} = \int e^{-x^2} dx = \frac{\sqrt{\pi}}{2} Erf(x) + C$

$y(1) = 1 \Rightarrow$

$e^{-1} = \frac{\sqrt{\pi}}{2} Erf(1) + C$. Hence

$y e^{-x^2} = \frac{\sqrt{\pi}}{2}(Erf(x) - Erf(1)) + e^{-1}$

$\Rightarrow y = \frac{\sqrt{\pi}}{2}(Erf(x) - Erf(1)) e^{x^2} + e^{x^2-1}$.

5) $(4xy^2-5)\,dx + (4x^2y+2)\,dy = 0$

$M = 4xy^2-5$; $N = 4x^2y+2$

$\frac{\partial M}{\partial y} = 8xy$; $\frac{\partial N}{\partial x} = 8xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so DE is Exact.

Now $\frac{\partial f}{\partial x} = M = 4xy^2-5$ — (1)

$\frac{\partial f}{\partial y} = N = 4x^2y+2$ — (2)

Integrate (1) w.r.t x

$f(x,y) = 2x^2y^2 - 5x + g(y)$

Diff partially w.r.t. y

$\frac{\partial f}{\partial y} = 4x^2y + g'(y)$ — (3)

Compare (2) and (3)

$g'(y) = 2 \Rightarrow g(y) = 2y$

Hence $f(x,y) = 2x^2y^2 - 5x + 2y = C$ is the solution

8) $(1+\ln x + \frac{y}{x})\,dx = (1-\ln x)\,dy$

Re-write as

$(1+\ln x+\frac{y}{x})\,dx - (1-\ln x)\,dy = 0$

$M = 1+\ln x + \frac{y}{x}$

$N = -1+\ln x$

$\frac{\partial M}{\partial y} = \frac{1}{x}$; $\frac{\partial N}{\partial x} = \frac{1}{x}$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, D.E. is exact

Now $\frac{\partial f}{\partial x} = M = 1+\ln x + \frac{y}{x}$ — (1)

$\frac{\partial f}{\partial y} = N = -1+\ln x$ — (2)

Integrate (1) w.r.t. x

$f(x,y) = \int (1+\ln x + \frac{y}{x})\,dx$

$= x + y\ln x + \int \ln x\,dx$

$= \cancel{x} + y\ln x + x\ln x - \cancel{x} + g(y)$

$= y\ln x + x\ln x + g(y)$

Diff w.r.t. y,

$\frac{\partial f}{\partial y} = \ln x + g'(y)$ — (3)

Compare (2) and (3)

$g'(y) = -1 \Rightarrow g(y) = -y$

Thus $f(x,y) =$

$y\ln x + x\ln x - y = C$

15) $(x^2y^3 - \frac{1}{1+9x^2})\frac{dx}{dy} + x^3y^2 = 0$

Re-write as

$(x^2y^3 - \frac{1}{1+9x^2})\,dx + x^3y^2\,dy = 0$

$M = x^2y^3 - \frac{1}{1+9x^2}$

$N = x^3y^2$

$\frac{\partial M}{\partial y} = 3x^2y^2$; $\frac{\partial N}{\partial x} = 3x^2y^2$

$\therefore$ D.E. is exact

Now $\frac{\partial f}{\partial x} = M = x^2y^3 - \frac{1}{1+9x^2}$ —(1)

$\frac{\partial f}{\partial y} = N = x^3y^2$ —(2)

Integrate (2) w.r.t. y

$f(x,y) = \frac{1}{3}x^3y^3 + h(x)$

$\frac{\partial f}{\partial x} = x^2y^3 + h'(x)$ — (3)

From (1) and (3) $\Rightarrow h'(x) = -\frac{1}{1+9x^2}$

$\Rightarrow h(x) = -\frac{1}{3}\tan^{-1}(3x)$

Hence $f(x,y) = C$ is solution

(2.4) contd

25) $(y^2\cos x - 3x^2y - 2x)\,dx + (2y\sin x - x^3 + \ln y)\,dy = 0$

$y(0) = e.$

$M = y^2\cos x - 3x^2y - 2x$

$N = 2y\sin x - x^3 + \ln y$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2y\cos x - 3x^2\\ \frac{\partial N}{\partial x} &= 2y\cos x - 3x^2\end{aligned}\right\}\text{Exact}$$

Now $\dfrac{\partial f}{\partial x} = y^2\cos x - 3x^2y - 2x$ —(1)

$\dfrac{\partial f}{\partial y} = 2y\sin x - x^3 + \ln y$ —(2)

Integrate (1) w.r.t. x

$f(x,y) = y^2\sin x - x^3y - x^2 + g(y)$

Diff. w.r.t. y

$\dfrac{\partial f}{\partial y} = 2y\sin x - x^3 + g'(y)$ —(3)

From (2) and (3) $g'(y) = \ln y$

$g(y) = \int \ln y\,dy = y\ln - y$

Hence

$f(x,y) = y^2\sin x - x^3y - x^2 + y\ln y - y = C$

27) Find k so that D.E is exact.

$(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

$M = y^3 + kxy^4 - 2x$

$N = 3xy^2 + 20x^2y^3$

$\dfrac{\partial M}{\partial y} = 3y^2 + 4kxy^3$ —(1)

$\dfrac{\partial N}{\partial x} = 3y^2 + 40xy^3$ —(2)

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ if

$4k = 40$ or $k = 10.$

For this value of k the DE is exact.

31) $(2y^2 + 3x)\,dx + 2xy\,dy = 0$

$$\left.\begin{aligned}M &= 2y^2 + 3x\\ N &= 2xy\end{aligned}\right\}\quad \frac{\partial M}{\partial y} = 4y,\quad \frac{\partial N}{\partial x} = 2y$$

D.E. is not exact. However if we multiply by x throughout

$(2y^2x + 3x^2)\,dx + 2x^2y\,dy = 0$

$M = 2y^2x + 3x^2$; $\dfrac{\partial M}{\partial y} = 4xy$

$N = 2x^2y$; $\dfrac{\partial N}{\partial x} = 4xy$

Now D.E. is exact.

$\dfrac{\partial f}{\partial x} = 2y^2x + 3x^2$ —(1)

$\dfrac{\partial f}{\partial y} = 2x^2y$ —(2)

Integrate (1) w.r.t. x

$f(x,y) = y^2x^2 + x^3 + g(y)$

$\Rightarrow \dfrac{\partial f}{\partial y} = 2yx^2 + g'(y)$ —(3)

From (2) & (3) $g'(y) = 0$

$\Rightarrow g(y) = 0$

Hence

$f(x,y) = x^2y^2 + x^3 = C.$