#11) $X' = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} X + \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $\qquad$ 8.3

$$\left.\begin{matrix} \lambda_1 = 0 & K_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 = 1 & K_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{matrix}\right\} \Rightarrow \begin{matrix} X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ X_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t \end{matrix}$$

$$\Phi(t) = \begin{bmatrix} 1 & 3e^t \\ 1 & 2e^t \end{bmatrix}$$

$X_p = \Phi \cdot U$ when $U = \int \Phi^{-1} F \, dt$

$|\Phi| = 2e^t - 3e^t = -e^t$, $\Phi^{-1} = \frac{1}{|\Phi|} \begin{bmatrix} 2e^t & -3e^t \\ -1 & 1 \end{bmatrix}$

$\Phi^{-1} = \begin{bmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{bmatrix}$, $\Phi^{-1} F = \begin{bmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -11 \\ 5e^{-t} \end{bmatrix}$

$$U = \int \begin{bmatrix} -11 \\ 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} -11t \\ -5e^{-t} \end{bmatrix}$$

Now, $X_p = \Phi \cdot U = \begin{bmatrix} 1 & 3e^t \\ 1 & 2e^t \end{bmatrix} \begin{bmatrix} -11t \\ -5e^{-t} \end{bmatrix} = \begin{bmatrix} -11t - 15 \\ -11t - 10 \end{bmatrix}$

The general sol is : $X = X_c + X_p$

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} -11t - 15 \\ -11t - 10 \end{bmatrix}$$

8.3

#23) $X' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} X + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t$

$P(\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda = \frac{2 \mp \sqrt{4-8}}{2} = \frac{2 \mp 2i}{2} = 1 \mp i$

eigenvector: $\left[\begin{array}{cc|c} 1-(1+i) & -1 & 0 \\ 1 & 1-(1+i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right] \xrightarrow{iR_1 + R_2} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow k_1 - ik_2 = 0$

$\Rightarrow k_1 = ik_2$

$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} ik_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$

$K_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ eigenvector for A assoc $\lambda = 1+i$

$\alpha = 1$
$\beta = 1$

$K_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_2} i$

$X_1 = (B_1 \cos t - B_2 \sin t) e^t = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t$

$X_2 = (B_2 \cos t + B_1 \sin t) e^t = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t$

Hence, $\Phi(t) = \begin{bmatrix} -e^t \sin t & e^t \cos t \\ e^t \cos t & e^t \sin t \end{bmatrix}$

Now, $X_p = \Phi(t)\, U(t)$

when $U(t) = \int \Phi^{-1}(t) F(t)\, dt$

$$|\Phi(t)| = -e^{2t}\sin^2 t - e^{2t}\cos^2 t = -e^{2t}$$

$$\Phi^{-1}(t) = \frac{1}{|\Phi(t)|}\begin{bmatrix} e^t \sin t & -e^t \cos t \\ -e^t \cos t & -e^t \sin t \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$

$$\Phi^{-1}(t)\,F(t) = \begin{bmatrix} -e^{-t}\sin t & e^{-t}\cos t \\ e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}\begin{bmatrix} e^t \cos t \\ e^t \sin t \end{bmatrix}$$

$$= \begin{bmatrix} -\sin t\cos t + \sin t \cos t \\ \cos^2 t + \sin^2 t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U(t) = \int \Phi^{-1}(t)\,F(t)\,dt = \int \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Therefore $X_p = \Phi(t)\,U(t) = \begin{bmatrix} -e^t \sin t & e^t \cos t \\ e^t \cos t & e^t \sin t \end{bmatrix}\begin{bmatrix} 0 \\ t \end{bmatrix}$

$$= \begin{bmatrix} t e^t \cos t \\ t e^t \sin t \end{bmatrix}$$

The general solution is

$$X(t) = \Phi(t)\,C + X_p$$

$$X(t) = c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} e^t + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^t + \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} t e^t$$