25) $A = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

I $\det(A - \lambda I) = 0 \Rightarrow$

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = 0$$

$\Rightarrow (5-\lambda)\left[-\lambda(5-\lambda)-4\right] + 4\left[5-\lambda\right] = 0$

$(5-\lambda)\left[\lambda^2 - 5\lambda - \cancel{4} + \cancel{4}\right] = 0 \Rightarrow \lambda = 0, 5, 5$

$\underline{\underline{\lambda = 0}}$ $\quad (A - \lambda I)\big|_{\lambda=0} \Rightarrow$

$$\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{\frac{1}{5}R_1} \begin{bmatrix} 1 & -4/5 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -\frac{4}{5} & 0 \\ 0 & \frac{4}{5} & 2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{\frac{5}{4}R_2} \begin{bmatrix} 1 & -\frac{4}{5} & 0 \\ 0 & 1 & \frac{5}{2} \\ 0 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 - 2R_2]{R_1 + \frac{4}{5}R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_3 &= \alpha \\ x_2 &= -\tfrac{5}{2}\alpha \\ x_1 &= -2\alpha \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{5}{2} \\ 1 \end{pmatrix}\alpha \Rightarrow K_1 = \begin{pmatrix} -2 \\ -5/2 \\ 1 \end{pmatrix}$$

$(\alpha = 1)$.

$\underline{\underline{\lambda = 5}}$: $\quad (A - \lambda I)\big|_{\lambda=5} \Rightarrow$

$$\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -5 & 2 \\ 0 & -4 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$-\frac{1}{4}R_2 \Rightarrow \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow[R_3 - 2R_2]{R_1 + 5R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $x_3 = t, \quad x_2 = 0, \quad x_1 = -2t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} t \quad \Rightarrow K_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

$\lambda = 5$ has defect 1.

To find P.

$(A - 5I)P = K_2 \Rightarrow$

$$\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -4P_2 &= -2 \\ P_1 - 5P_2 + 2P_3 &= 0 \\ 2P_2 &= 1 \end{aligned}\right\} P_2 = \frac{1}{2}$$

This gives $P_1 + 2P_3 = 5P_2 = \frac{5}{2}$

If we take $P_3 = 1$, $\quad P_1 = \frac{5}{2} - 2 = \frac{1}{2}$

$$P = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Hence the solution can be written as

$$X = C_1 \begin{pmatrix} -2 \\ -5/2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{5t} + C_3 \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} t e^{5t} + \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} \right\}.$$