45) $A = \begin{pmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix}$

I $\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} 1-\lambda & -12 & -14 \\ 1 & 2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$

After simplification

$\Rightarrow (1-\lambda)(\lambda^2+25)=0$

$\Rightarrow \lambda = 1, \ \lambda = \pm 5i$

II $\lambda=1$, $(A-\lambda I)\big|_{\lambda=1}$ gives

$\begin{bmatrix} 0 & -12 & -14 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{bmatrix}$ ① $R_1 \leftrightarrow R_2$ ② $R_3 - R_1$ ③ $-\frac{1}{12} R_2$ ④ $R_1 - R_2$

$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{25}{6} \\ 0 & 1 & \frac{7}{6} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{pmatrix} \frac{25}{6}\alpha \\ -\frac{7}{6}\alpha \\ \alpha \end{pmatrix}$

For $\alpha = 6$ $K_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}$.

$\boxed{X_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t}$

Next, $\lambda = 5i$, $(A-\lambda I)\big|_{\lambda=5i}$

$= \begin{pmatrix} 1-5i & -12 & -14 \\ 1 & 2-5i & -3 \\ 1 & 1 & -2-5i \end{pmatrix}$

Then $R_1 \leftrightarrow R_3$, $R_2 - R_1$

$\frac{1}{1-5i} R_2$, $R_3 + (13-5i) R_2$

and $R_1 - R_2$ lead to

$\begin{pmatrix} 1 & 0 & -1-5i \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

This gives $x_3 = t$, $x_2 = t$

$x_1 = 1+5i$

So $K = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin t$

$X_3 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin t$

$X = C_1 X_1 + C_2 X_2 + C_3 X_3 =$

$\begin{pmatrix} C_1(25)e^t + C_2\cos t - C_2 5\sin t + C_3 5\cos t + C_3 \sin t \\ C_1(-7)e^t + C_2\cos 5t + C_3 \sin 5t \\ C_1(6)e^t + C_2 \cos 5t + C_3 \sin 5t \end{pmatrix}$

Using $X(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$

We get

$$\left.\begin{aligned} 25C_1 + C_2 + 5C_3 &= 4 \\ -7C_1 + C_2 &= 6 \\ 6C_1 + C_2 &= -7 \end{aligned}\right\} \begin{aligned} &-(1) \\ &-(2) \\ &-(3) \end{aligned}$$

Solving (2), (3) $C_1 = -1$, $C_2 = -1$

From ① $-25 - 1 + 5C_3 = 4$

$\Rightarrow C_3 = 6.$

Hence the solution can be written as

$-\begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos t + \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \sin 5t$

$+ 6 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cos 5t + 6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \sin 5t.$