6) $y'' + y = \sec^2 x$

Step 1 Homogeneous D E:

$$y'' + y = 0$$

Auxiliary eqn: $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

Step 2: $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} = \sin x \sec^2 x = \tan x \sec x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} = \sec x$$

Thus $u_1' = \tan x \sec x \Rightarrow u_1 = \sec x$

$u_2' = \sec x \Rightarrow u_2 = \ln|\sec x + \tan x|$

Hence $y_p = \sec x \cos x + \sin x \ln|\sec x + \tan x|$

$= 1 + \sin x \ln|\sec x + \tan x|$.

Step 3 $y = y_c + y_p$

$= C_1 \cos x + C_2 \sin x + 1 + \sin x \ln|\sec x + \tan x|$.

11) $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Step 1: Homogeneous D.E is

$$y'' + 3y' + 2y = 0$$

Auxiliary Eqn: $m^2 + 3m + 2 = 0$

$m = -1, -2$

$$y_1 = e^{-x}, \quad y_2 = e^{-2x}$$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Step 2: $y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ 1/(1+e^x) & -2e^{-2x} \end{vmatrix} = \frac{-e^{-2x}}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x}$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{-2x}}{(1+e^x)(-e^{-3x})} = \frac{e^x}{1+e^x}$$

$$u_1 = \int \frac{e^x}{1+e^x} dx = \ln(1+e^x)$$

$$u_2' = \frac{W_2}{W} = \frac{e^{-x}}{(1+e^x)(-e^{-3x})} = \frac{-e^{2x}}{1+e^x}$$

$$u_2 = \int -\frac{e^{2x}}{1+e^x} dx = \int \left(\frac{e^x}{e^x+1} - e^x\right) dx$$

(by division)

$$= \ln(1+e^x) - e^x.$$

Thus $y_p = e^{-x} \ln(1+e^x) + e^{-2x} \left[\ln(1+e^x) - e^x\right]$

$$= e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) + e^{-x}$$

Step 3 $y = y_c + y_p$

$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) + e^{-x}$

or $y = C_1' e^{-x} + C_2 e^{-2x} + (e^{-x} + e^{-2x}) \ln(1+e^x)$

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4.6 cont.

13) $y'' + 3y' + 2y = \sin e^{x}$

Step1 Homogeneous D.E. is

$$y'' + 3y' + 2y = 0$$

Auxiliary equation: $m^2 + 3m + 2 = 0$

$$m = -1, -2$$

$$y_1 = e^{-x}, \quad y_2 = e^{-2x}$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step2. $y_p = u_1 y_1 + u_2 y_2$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^{x} & -2e^{-2x} \end{vmatrix} = -e^{-2x} \sin e^{x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^{x} \end{vmatrix} = e^{-x} \sin e^{x}$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{-2x} \sin e^{x}}{-e^{-3x}} = e^{x} \sin e^{x}$$

$$u_1 = \int e^{x} \sin e^{x}\, dx = -\cos(e^{x}).$$

$$u_2' = \frac{W_2}{W} = \frac{e^{-x} \sin e^{x}}{-e^{-3x}} = -e^{2x} \sin e^{x}$$

$$u_2 = \int -e^{2x} \sin e^{x}\, dx \qquad \text{Put } e^{x} = t, \; e^{x} dx = dt$$

$$u_2 = \int -t \sin t\, dt = t \cos t - \sin t$$

$$= e^{x} \cos(e^{x}) - \sin(e^{x})$$

Hence $y_p = \cancel{-e^{-x} \cos(e^{x})} + \cancel{e^{-2x} e^{x} \cos(e^{x})} - e^{-2x} \sin(e^{x})$

$$= -e^{-2x} \sin(e^{x})$$

Step3

$$y = y_c + y_p$$

24) Use Cauchy-Euler Method for y_c and then same step2 as above.

25) $y''' + y' = \tan x$

Step1 Homogeneous D.E. is

$$y''' + y' = 0$$

Auxiliary Eqn: $m^3 + m = 0$

$$m = 0, \pm i$$

$$y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x$$

$$y_c = c_1 + c_2 \cos x + c_3 \sin x$$

Step2 $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1.$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x$$

$$u_1' = \frac{W_1}{W} = \tan x \Rightarrow u_1 = \ln|\sec x|$$

$$u_2' = \frac{W_2}{W} = -\sin x \Rightarrow u_2 = \cos x$$

$$u_3' = \frac{W_3}{W} = -\sin x \tan x = -\frac{\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x - 1}{\cos x} \qquad u_3 \Rightarrow \int (\cos x - \sec x)\, dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

Thus

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = C_1 + C_2 \cos x + C_3 \sin x + \ln|\sec x| - \sin x \ln|\sec x + \tan x|$$

(After simplification)