King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260 Exam-II
Semester II, 2010- (092)

Name:	KEY		
ID:	KEY	**Serial no:**	

Section

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7:00-7:50	8:00-8:50	9:00-9:50	10:00-10:50
Dr. Fairag	Dr. Fairag	Dr. Laradji	Dr. Fairag

Q	FORM: A	Points
1		10
2		10
3		10
4		10
5		13
6		10
7		14
8		10
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Total		100

☺ Say Bismillah & Good luck ☺

(1) Use <u>Cayley-Hamilton</u> theorem to compute $A^4 - 3A^3$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix}$. **(show all your work)**

The characteristic polynomial $P(\lambda) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -1 \\ -2 & -3 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -3 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ -2 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 1-\lambda \\ -2 & -3 \end{vmatrix}$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 2\right] - 2\left[-2\lambda\right] - \left[-4 - 2\lambda\right]$$

$$= -\lambda^3 + 2\lambda^2 + 2\lambda + \lambda^2 - 2\lambda - 2 + 4\lambda + 4 + 2\lambda$$

$$= -\lambda^3 + 3\lambda^2 + 6\lambda + 2. \quad 3$$

By Cayley-Hamilton theorem

$$-A^3 + 3A^2 + 6A + 2I = 0 \quad 2$$

or $A^3 - 3A^2 = 6A + 2I = 6\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 12 & -6 \\ 12 & 8 & -6 \\ -12 & -18 & 8 \end{bmatrix} \quad 2$$

Now, $A^4 - 3A^3 = A(A^3 - 3A^2) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 8 & 12 & -6 \\ 12 & 8 & -6 \\ -12 & -18 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 44 & 46 & -26 \\ 40 & 50 & -26 \\ -64 & -66 & 38 \end{bmatrix} \quad 3$$

(2) Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

(show all your work)

$$A \rightarrow \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the leading columns are the 1^{st}, 3^{rd} and 5^{th}, hence a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

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(3) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 2 & 3 & 6 & 7 \\ 2 & 1 & 3 & 6 & 5 \end{bmatrix}$$

(show all your work)

$$A \longrightarrow \begin{matrix} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 5 & 6 & 7 \\ 0 & -1 & 7 & 6 & 5 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 4 & 6 & 7 \\ 0 & 0 & 12 & 12 & 12 \end{matrix}$$

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There are 3 nonzero rows in this echelon form matrix, hence rank $A = 3$.

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(4) Determine whether or not the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable. **(show all your work)**

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (-2-\lambda)^2(1-\lambda) = (2+\lambda)^2(1-\lambda). \quad 2$$

The eigenvalues of A are $-2, -2, 1$. 3

For the eigenvalue -2, we get the homogeneous system

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ which gives $x + 3y = 0$, $z = 0$.

Hence a basis of the solution space of this homogeneous system is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$. Hence the dimension of the eigenspace

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E_{-2} is 1 which is less than the multiplicity of $\lambda = -2$.

So the matrix is not diagonalizable.

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(5) The characteristic equation of the matrix $C = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$

is $(2-x)^2(3-x)$.

Find the Jordan form of the matrix C. **(show all your work)**

The eigenvalues of C are $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = 3$.

So the Jordan form of C is

3

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We compute the dimension of the eigenspace E_2.

The homogeneous system $\begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $x+y-2z=0$ and $-x-2y+5z=0$. If we put $z=t$, we obtain $x+y=2t$ & $x+2y=5t$, i.e. $x=-t$ & $y=3t$.

So a basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right\}$. This means that C is not diagonalizable ($\because \dim E_2 = 1 <$ multiplicity of λ_2).

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Hence Jordan form of C is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

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(6) Given that the matrix $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$ is diagonalizable and has eigenvalues $\lambda = -3, 1, 1$.

Find a diagonalizing matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(show all your work)

$\underline{\lambda = -3}$ Homogeneous system $\begin{bmatrix} 8 & -4 & 4 \\ 12 & -8 & 12 \\ 4 & -4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $2x - y + z = 0$ & $x - 2y + 2z = 0$, with a basis for the solution space $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$. 2

$\underline{\lambda = 1}$ Homogeneous system $\begin{bmatrix} 4 & -4 & 4 \\ 12 & -12 & 12 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $x - y + z = 0$, with a basis for the solution space $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. 4

The required matrices are

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2 2

(7) True or False.

[a]	The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^5+3\frac{d^2y}{dx^2}-9\left(\frac{d^4y}{dx^4}\right)^2-y=0$ is 5.	(T) (F)
[b]	The equation $\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial u}{\partial y}=y\sin x$ is a partial differential equation.	(T) (F)
[c]	$y(x)=0$ is a singular solution for the differential equation $y^{-1/2}\frac{dy}{dx}=x$.	(T) (F)
[d]	$y(x)=0$ is a singular solution for $y'=xy^{1/2}$.	(T) (F)
[e]	Let A be 4×4 matrix. A has eigenvalues $\lambda=2,2,5,7$, then Rank(A) $\neq 4$.	(T) (F)
[f]	$y(x)=x+\ln x$ is a particular solution of the differential equation $x^2y''+xy'-y=\ln x$.	(T) (F)
[g]	Let A be 4×4 matrix with characteristic polynomial $p(\lambda)=\lambda(\lambda-2)^3$, then det(A) $=0$.	(T) (F)

(8) Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$. **(show all your work)**

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 1 \\ 3 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \quad C_1 + C_2$$

$$= \begin{vmatrix} -\lambda & 2-\lambda & 1 \\ 3 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} \quad \text{expand third row}$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 2-\lambda & 1 \end{vmatrix} + (1-\lambda)\begin{vmatrix} -\lambda & 2-\lambda \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) - (2-\lambda) + (1-\lambda)\left[\lambda^2 - 2\lambda - 6 + 3\lambda\right]$$

$$= (1-\lambda)\left[\lambda^2 + \lambda - 6\right] = (1-\lambda)(\lambda+3)(\lambda-2)$$

Hence, $P(\lambda) = (1-\lambda)(\lambda+3)(\lambda-2)$ 4

$\lambda = 1, 2, -3$ are eigenvalues of A.

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(9) Solve the separable differential equation

$2(y^2+2y)\dfrac{dy}{dx} = (3x^2-4)(y^2-1)(y+2)$. **(show all your work)**

We divid both sides of the differential equation by

$(y^2-1)(y+2)$ to get

$$\frac{2y(y+2)}{(y^2-1)(y+2)}\frac{dy}{dx} = (3x^2-4)$$

Separation of variables gives

$$\int \frac{2y}{(y^2-1)}\,dy = \int (3x^2-4)\,dx \quad \boxed{3}$$

Aside: $u = y^2-1$

$du = 2y\,dy$

$\displaystyle\int \frac{2y\,dy}{y^2-1} = \int \frac{du}{u}$

$= \ln|u| = \ln|y^2-1|$

$$\ln|y^2-1| = x^3-4x+C_1 \quad \boxed{3}$$

$$|y^2-1| = e^{C_1}\cdot e^{x^3-4x}$$

$$y^2-1 = (\pm e^{C_1})(e^{x^3-4x})$$

$$y^2 = 1+Ae^{x^3-4x} \quad \boxed{3}$$

The value $A=0$ gives the solutions $y(x)=1$, $y(x)=-1$,

but no value of A gives the singular solution $y(x) = -2$

$\boxed{4}$

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