

King Fahd University of Petroleum and Minerals
 Department of Mathematics and Statistics
 Math 260 Exam-II
 Semester II, 2010- (092)

Name:	KEY		
ID:	KEY	Serial no:	

Section

1
7:00-7:50
Dr. Fairag

2
8:00-8:50
Dr. Fairag

3
9:00-9:50
Dr. Laradji

4
10:00-10:50
Dr. Fairag

Q	FORM: A	Points
1		10
2		10
3		10
4		10
5		13
6		10
7		14
8		10
9		13
Total		100

☺ Say Bismillah & Good luck ☺

(1) Use Cayley-Hamilton theorem to compute $A^4 - 3A^3$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix}$. (show all your work)

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -1 \\ -2 & -3 & 1-\lambda \end{vmatrix}$$

The characteristic polynomial $P(\lambda) =$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -3 & 1-\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ -2 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 1-\lambda \\ -2 & -3 \end{vmatrix}$$

$$= (1-\lambda) [\lambda^2 - 2\lambda - 2] - 2 [-2\lambda] - [-4 - 2\lambda]$$

$$= -\lambda^3 + 2\lambda^2 + 2\lambda + \lambda^2 - 2\lambda - 2 + 4\lambda + 4 + 2\lambda$$

$$= -\lambda^3 + 3\lambda^2 + 6\lambda + 2$$

By Cayley-Hamilton theorem

$$-A^3 + 3A^2 + 6A + 2I = 0$$

$$\text{or } A^3 - 3A^2 = 6A + 2I = 6 \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 & -6 \\ 12 & 8 & -6 \\ -12 & -18 & 8 \end{bmatrix}$$

$$\text{Now, } A^4 - 3A^3 = A(A^3 - 3A^2) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 8 & 12 & -6 \\ 12 & 8 & -6 \\ -12 & -18 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 46 & -26 \\ 40 & 50 & -26 \\ -64 & -66 & 38 \end{bmatrix}$$

(2) Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

(show all your work)

$$A \rightarrow \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the leading columns are the 1st, 3rd and 5th, hence a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

(3) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 2 & 3 & 6 & 7 \\ 2 & 1 & 3 & 6 & 5 \end{bmatrix}$$

(show all your work)

$$A \rightarrow \begin{array}{ccccc} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 4 & 6 & 7 \\ 0 & -1 & 7 & 6 & 5 \end{array} \rightarrow \begin{array}{ccccc} 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 4 & 6 & 7 \\ 0 & 0 & 11 & 12 & 12 \end{array}$$

There are 3 ~~non-zero~~ non-zero rows in this echelon form matrix, hence $\text{rank } A = 3$.

(4) Determine whether or not the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable. (show all your work)

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda) \begin{vmatrix} -2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} \\ &= (-2-\lambda)^2(1-\lambda) = (2+\lambda)^2(1-\lambda). \end{aligned}$$

The eigenvalues of A are $-2, -2, 1$.

For the eigenvalue -2 , we get the homogeneous system

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ which gives } x+3y=0, z=0.$$

Hence a basis of the solution space of this homogeneous system is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$. Hence the dimension of the eigenspace

E_{-2} is 1 which is less than the multiplicity of $\lambda = -2$.

So the matrix is not diagonalizable.

(5) The characteristic equation of the matrix $C = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{bmatrix}$

is $(2-x)^2(3-x)$.

Find the Jordan form of the matrix C . (show all your work)

The eigenvalues of C are $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3$.
So the Jordan form of C is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We compute the dimension of the eigenspace E_2 .

The homogeneous system $\begin{bmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $x+y-2z=0$ and $-x-2y+5z=0$. If we put $z=t$, we obtain $x+y=2t$ & $x+2y=5t$, i.e. $x=-t$ & $y=3t$.

So a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$. This means that C is not diagonalizable ($\because \dim E_2 = 1 < \text{multiplicity of } \lambda_2$).

Hence Jordan form of C is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(6) Given that the matrix $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$ is diagonalizable and has eigenvalues $\lambda = -3, 1, 1$.

Find a diagonalizing matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(show all your work)

$\lambda = -3$ Homogeneous system $\begin{bmatrix} 8 & -4 & 4 \\ 12 & -8 & 12 \\ 4 & -4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $2x - y + z = 0$ & $x - 2y + 2z = 0$, with a basis for the solution space $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 1$ Homogeneous system $\begin{bmatrix} 4 & -4 & 4 \\ 12 & -12 & 12 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

gives $x - y + z = 0$, with a basis for the solution space $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The required matrices are

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(7) True or False.

[a]	The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^5 + 3\frac{d^2y}{dx^2} - 9\left(\frac{d^4y}{dx^4}\right)^2 - y = 0$ is 5.	(T) (F)
[b]	The equation $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} = y \sin x$ is a partial differential equation.	(T) (F)
[c]	$y(x) = 0$ is a singular solution for the differential equation $y^{-1/2} \frac{dy}{dx} = x$.	(T) (F)
[d]	$y(x) = 0$ is a singular solution for $y' = xy^{1/2}$.	(T) (F)
[e]	Let A be 4×4 matrix. A has eigenvalues $\lambda = 2, 2, 5, 7$, then $\text{Rank}(A) \neq 4$.	(T) (F)
[f]	$y(x) = x + \ln x$ is a particular solution of the differential equation $x^2 y'' + xy' - y = \ln x$.	(T) (F)
[g]	Let A be 4×4 matrix with characteristic polynomial $p(\lambda) = \lambda(\lambda - 2)^3$, then $\det(A) = 0$.	(T) (F)

(8) Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{(show all your work)}$$

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 1 \\ 3 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} \quad C_1 + C_2$$

$$= \begin{vmatrix} -\lambda & 2-\lambda & 1 \\ 3 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} \quad \text{expand third row}$$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 2-\lambda & 1 \end{vmatrix} + (1-\lambda) \begin{vmatrix} -\lambda & 2-\lambda \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda) - (2-\lambda) + (1-\lambda) [\lambda^2 - 2\lambda - 6 + 3\lambda]$$

$$= (1-\lambda) [\lambda^2 + \lambda - 6] = (1-\lambda)(\lambda+3)(\lambda-2)$$

$$\text{Hence, } P(\lambda) = (1-\lambda)(\lambda+3)(\lambda-2)$$

$\lambda = 1, 2, -3$ are eigenvalues of A .

(9) Solve the separable differential equation

$$2(y^2 + 2y) \frac{dy}{dx} = (3x^2 - 4)(y^2 - 1)(y + 2) \quad \text{(show all your work)}$$

We divide both sides of the differential equation by $(y^2 - 1)(y + 2)$ to get

$$\frac{2y(y+2)}{(y^2-1)(y+2)} \frac{dy}{dx} = (3x^2-4)$$

Separation of variables gives

$$\int \frac{2y}{(y^2-1)} dy = \int (3x^2-4) dx \quad \triangle 3$$

$$\ln |y^2-1| = x^3 - 4x + C_1 \quad \triangle 3$$

$$|y^2-1| = e^{C_1} \cdot e^{x^3-4x}$$

$$y^2-1 = (\pm e^{C_1}) (e^{x^3-4x})$$

$$y^2 = 1 + A e^{x^3-4x} \quad \triangle 3$$

The value $A=0$ gives the solutions $y(x)=1, y(x)=-1$,
but no value of A gives the singular solution $y(x)=-2$

$\triangle A$

Aside: $u = y^2 - 1$
 $du = 2y dy$
 $\int \frac{2y dy}{y^2-1} = \int \frac{du}{u}$
 $= \ln |u| = \ln |y^2-1|$

