King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 260 Exam-1
Semester II, 2010- (092)

Name:	KEY		
ID:	KEY	**Serial no:**	

Section

1	2	3	4
7:00-7:50	8:00-8:50	9:00-9:50	10:00-10:50
Dr. Fairag	Dr. Fairag	Dr. Laradji	Dr. Fairag

Q	FORM: **A**	Points
1		**12**
2		**9**
3		**10**
4		**10**
5		**9**
6		**22**
7		**7**
8		**7**
9		**7**
10		**7**
Total		**100**

☺ Say Bismillah & Good luck ☺

(1) Consider the homogeneous linear system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 + 3x_4 \quad &= 0 \\ -x_1 \quad - 2x_3 - x_4 + x_5 &= 0 \\ 2x_1 + 3x_2 + 7x_3 + 8x_4 + x_5 &= 0 \end{aligned}$$

(a) Find the solution space W.

(b) Find a basis for W.

(c) Find dim(W).

The augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 3 & 0 & 0 \\ -1 & 0 & -2 & -1 & 1 & 0 \\ 2 & 3 & 7 & 8 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leading variables: x_1, x_2 4

Free variables: x_3, x_4, x_5

we set $x_3 = t_1$, $x_4 = t_2$, $x_5 = t_3$

The reduced system yields

$$x_2 = -x_3 - 2x_4 - x_5$$
$$= t_1 - 2t_2 - t_3$$

$$x_1 = -2x_3 - x_4 + x_5$$
$$= -2t_1 - t_2 + t_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} t_1 + \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} t_2 + \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3} t_3$$

Thus the solution space of the system is a 3-dim subspace of R^5 with basis $\{v_1, v_2, v_3\}$ 2 6

(2) Given that: $u = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $t = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$,

Express u as a linear combination of the vectors v, w, t.

we write the equation $u = c_1 v + c_2 w + c_3 t$.

The augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 3 \\ 1 & 1 & 5 & 3 \\ -1 & 5 & 3 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 5 & 3 \\ 4 & 1 & 2 & 3 \\ -1 & 5 & 3 & 3 \end{array}\right] \xrightarrow[R_1 + R_3]{-4R_1 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & 3 \\ 0 & -3 & -18 & -9 \\ 0 & 6 & 8 & 6 \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 5 & 3 \\ 0 & 1 & 6 & 3 \\ 0 & 3 & 4 & 3 \end{array}\right] \xrightarrow{-3R_2 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & 3 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -14 & -6 \end{array}\right]$$

3

we see now, from the third row

$$c_3 = -6/-14 = 3/7$$

and second row and first row

$$c_2 = 3 - 6c_3 = 3 - 18/7 = 3/7$$

$$c_1 = 3 - c_2 - 5c_3 = 3 - 3/7 - 15/7 = 3/7$$

we have found $\boxed{c_1 = 3/7, \quad c_2 = 3/7, \quad c_3 = 3/7}$ 6

$$u = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \left(\frac{3}{7}\right)\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + \left(\frac{3}{7}\right)\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + \left(\frac{3}{7}\right)\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

(3) Compute A^{-1} if $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1 + R_3}$$ 2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{2R_2 + R_3}$$ 2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_1}$$ 2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] = [I \mid A^{-1}]$$

Now, we see that the inverse of A

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$ 4

(4) Let W be the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3 such that $x_3 = -2x_1$. Is W a subspace of R^3? (either prove it is or show that it is not a subspace of R^3).

If $u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors in W,

then $x_3 = -2x_1$ and $y_3 = -2y_1$.

Now, $u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ with

$x_3 + y_3 = -2x_1 - 2y_1$ which implies $x_3 + y_3 = -2(x_1 + y_1)$

So $u + v$ is also in W. (W closed under addition)

4

If c is a scalar, then

$cu = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ with

$cx_3 = c(-2x_1) = -2(cx_1)$

4

So cu is in W. (W closed under scalar multiplication)

Thus, we have shown that W satisfies conditions (i) and (ii) of Theorem 1 in section 4.2 and is therefore a subspace of R^3.

2

(5) Determine for what values of k the set **S** is linearly independent in R^4.

$$S = \left\{ \begin{array}{cccc} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \begin{bmatrix} 100 \\ 1 \\ 2 \\ -1 \end{bmatrix}, & \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} -9 \\ 1 \\ 1 \\ k \end{bmatrix} \\ v_1 & v_2 & v_3 & v_4 \end{array} \right\}$$

From section 4.3, The vectors v_1, v_2, v_3, v_4 in R^4 are linearly independent if and only if the determinant of the 4×4 matrix $A = [v_1\ v_2\ v_3\ v_4]$ is nonzero.

3

$$|A| = \begin{vmatrix} 1 & 100 & 3 & -9 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 0 & k \end{vmatrix} \underset{\text{the first Column}}{\overset{\text{expand along}}{=}} (1) \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 0 & k \end{vmatrix}$$

$$\overset{-C_1+C_3}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & k+1 \end{vmatrix} \underset{\text{first row}}{\overset{\text{expand along}}{=}} (1) \begin{vmatrix} 1 & -1 \\ 0 & k+1 \end{vmatrix} = (1)(k+1)$$

3

Now, $|A| = k+1 \neq 0 \Rightarrow \boxed{k \neq -1}$

3

(6) True or False. (2pts each)

[1]	The inverse of any invertible matrix is unique.	(T) (F)
[2]	The determinant of a matrix is equal to the determinant of its transpose.	(T) (F)
[3]	Any subset of a linearly independent set $S=\{v_1,v_2,\cdots,v_k\}$ is a linearly independent set of vectors.	(T) (F)
[4]	Let A be an nxn matrix. If **A** has nonzero determinant then **A** is row equivalent to the identity matrix **I**.	(T) (F)
[5]	Any set of more than n vectors in R^n is linearly dependent	(T) (F)
[6]	The vectors $v_1, v_2, \cdots, v_k$ are linearly dependent if and only if one of them is a linear combination of the others	(T) (F)
[7]	If u, v and w in R^3 are linearly independent, then they constitute a basis for R^3.	(T) (F)
[8]	The set $S=\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}\right\}$ is a basis for R^2	(T) (F)
[9]	The set $S=\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\end{bmatrix}\right\}$ is a basis for R^2	(T) (F)
[10]	The set $S=\left\{\begin{bmatrix}1\\0\\0\\5\end{bmatrix}, \begin{bmatrix}2\\1\\0\\3\end{bmatrix}, \begin{bmatrix}3\\4\\1\\9\end{bmatrix}\right\}$ is linearly independent in R^4	(T) (F)
[11]	The set $S=\left\{\begin{bmatrix}6\\3\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ spans R^2	(T) (F)

(7) Which of the following subsets of R^4 is linearly independent?

(a) $\left\{ \underset{v_1}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}2\\-1\\1\\-2\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}3\\1\\4\\2\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}3\\4\\1\\2\end{bmatrix}} \right\}$ $V_3 = V_1 + V_2$

(b) $\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}-1\\2\\-3\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\-3\\4\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \right\}$ V_4 zero vector

(c) $\left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\1\\4\\3\end{bmatrix}, \begin{bmatrix}3\\1\\4\\2\end{bmatrix}, \begin{bmatrix}3\\4\\1\\2\end{bmatrix}, \begin{bmatrix}1\\3\\2\\4\end{bmatrix} \right\}$ number of vectors $> \dim(R^4)$

(d) $\left\{ \underset{v_1}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}2\\1\\-2\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}1\\-1\\-5\\-4\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}4\\2\\-4\\0\end{bmatrix}} \right\}$ $V_4 = V_1 + V_2 + V_3$

(e) $\left\{ \underset{v_1}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}2\\3\\4\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}3\\4\\0\\0\end{bmatrix}}, \underset{v_4}{\begin{bmatrix}2\\0\\0\\0\end{bmatrix}} \right\}$ $\det([v_1\ v_2\ v_3\ v_4]) = (4)(4)(4)(2) \neq 0$

Correct : $\boxed{e}$

(8) The value of k for which the system

$$\begin{aligned} x+y+2z&=1\\ x-y+z&=2\\ -x-2y+z&=3\\ 2x-y+2z&=k \end{aligned}$$

has a unique solution is

(a) 5
(b) 4
(c) 3
(d) 2
(e) 1

$$\left[\begin{array}{ccc|c} 1&1&2&1\\ 1&-1&1&2\\ -1&-2&1&3\\ 2&-1&2&k \end{array}\right] \xrightarrow[-2R_1+R_4]{-R_1+R_2,\ R_1+R_3} \left[\begin{array}{ccc|c} 1&1&2&1\\ 0&-2&-1&1\\ 0&-1&3&4\\ 0&-3&-2&k-2 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{-R_3} \left[\begin{array}{ccc|c} 1&1&2&1\\ 0&1&-3&-4\\ 0&-2&-1&1\\ 0&-3&-2&k-2 \end{array}\right]$$

$$\xrightarrow[3R_2+R_4]{2R_2+R_3} \left[\begin{array}{ccc|c} 1&1&2&1\\ 0&1&-3&-4\\ 0&0&-7&-7\\ 0&0&-11&k-14 \end{array}\right] \xrightarrow{-\frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1&1&2&1\\ 0&1&-3&-4\\ 0&0&1&1\\ 0&0&-11&k-14 \end{array}\right] \xrightarrow{11R_3+R_4}$$

$$\left[\begin{array}{ccc|c} 1&1&2&1\\ 0&1&-3&-4\\ 0&0&1&1\\ 0&0&0&k-3 \end{array}\right]$$ to have a unique solution $\boxed{k=3}$

(9) The augmented coefficient matrix of a linear system of equations has reduced row echelon form

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

leading variables: x_1, x_3, x_4, x_6

Free variables: x_2, x_5

Then the number of free variables is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

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(10) $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{R_3+R_1}{=} \begin{vmatrix} 0 & 0 & -3 & -1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} \overset{-3C_4+C_3}{=}$

(a) -39
(b) -36
(c) 0
(d) 39
(e) 36

$\begin{vmatrix} 0 & 0 & 0 & -1 \\ 5 & -9 & -3 & 3 \\ -1 & 2 & 0 & -2 \\ 2 & 8 & 3 & 1 \end{vmatrix}$ $\underset{\text{the first row}}{\overset{\text{expand along}}{=}}$ $(-)(-1)\begin{vmatrix} 5 & -9 & -3 \\ -1 & 2 & 0 \\ 2 & 8 & 3 \end{vmatrix}$

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$= \begin{vmatrix} 5 & -9 & -3 \\ -1 & 2 & 0 \\ 2 & 8 & 3 \end{vmatrix} \overset{R_3+R_1}{=} \begin{vmatrix} 7 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 8 & 3 \end{vmatrix}$ $\underset{\text{third column}}{\overset{\text{expand along}}{=}}$

$+(3)\begin{vmatrix} 7 & -1 \\ -1 & 2 \end{vmatrix} = (3)[14-1] = 3(13) = 39$